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SYMMETRY TECHNIQUES FOR q-SERIES: ASKEY-WILSON POLYNOMIALS

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ABSTRACT. We advocate the exploitation of symmetry (recurrence relation) techniques for the derivation of properties associated with families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. Here these ideas are applied to the (continuous) Askey-Wilson polynomials, introduced by Askey and Wilson, to obtain a strikingly simple derivation of their orthogonality relations.

1. Introduction. In [**1**] the authors and A.K. Agarwal introduced symmetry techniques for the study of families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. The fundamental objects in this study are the recurrence relations obeyed by the families: generating functions and identities for each family are characterized in terms of the recurrence relations. The functions studied in [**1**] are of the form

$$ {}_r\varphi_s\begin{pmatrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{pmatrix}; q, z\Big) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k\cdots(a_r;q)_k\; z^k}{(b_1;q)_k\cdots(b_s;q)_k(q;q)_k} $$

and many-variable extension. Here

$$ (a;q)_k = \begin{cases} 1 & \text{if } k=0 \\ (1-a)(1-qa)\dots(1-q^{k-1}a) & \text{if } k=1,2,\dots \end{cases} $$

and $|q|<1$. In this paper we study the Askey-Wilson polynomials

$$ \phi_n^{(a,b,c,d)}(x) = {}_4\varphi_3\Big(\begin{matrix} q^{-n}, q^{n-1}abcd,\ az,\ a/z \\ ab,\ ac,\ ad \end{matrix}; q, q\Big) \tag{1.1} $$

where $n=0,1,2,\dots,$ a,b,c,d,q real and $(|a|,|b|,|c|,|d|)<1$. Initially we also require $abcd \neq 0$. These functions are orthogonal polynomials of order n in $x=(z+z^{-1})/2$ where frequently $z=e^{i\theta}$, $0\le\theta\le\pi$.

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