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BERNSTEIN INEQUALITIES IN L_p, $0 \leq p \leq +\infty$

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1. Introduction. The norm (or quasi-norm) in the space $L_p(T)$ of a function f is defined by

$$(1.1) \qquad ||f||_p = \left(\frac{1}{2\pi}\int_T |f(t)|^p\,dt\right)^{1/p}, \quad 0 < p < \infty.$$

The limiting cases are: for $p \to \infty$ the supremum norm $||f||_\infty$, and for $p \to 0$ (see [**3**, p. 139]) the quasi-norm of L_0,

$$||f||_0 = \exp\frac{1}{2\pi}\int_T \log|f(t)|\,dt.$$

For each of these spaces, one has the inequality

$$(1.2) \qquad \left|\left|\frac{1}{n}T_n'\right|\right|_p \leq ||T_n||_p, \quad 0 \leq p \leq +\infty,$$

where $T_n \in \mathcal{T}_n$, and $\mathcal{T}_n$ is the space of all trigonometric polynomials of degree $\leq n$, with complex coefficients. For $p = \infty$, the relation (1.2) is called the Bernstein inequality; for $1 \leq p < \infty$, it has been established by Zygmund, using an interpolation formula of M. Riesz. This case of (1.2) immediately follows from the Hardy-Littlewood-Pólya order relation $T_n' \prec nT_n$ established in Lorentz [**5**].

For $0 < p < 1$, the inequality (1.2) has been proved by Máté and Nevai [**4**] with an extra factor $(4e)^{1/p}$ on the right. A year later, Arestov [**1**] obtained (1.2) as it stands. The proofs of Máté and Nevai and of Arestov are complicated, and it is desirable to have simple proofs. We do so in §2; as a premium, we obtain a generalization of (1.2), which replaces the map $T_n \to \frac{1}{n}T_n'$ with a map $T_n \to AT_n + \frac{B}{n}T_n'$, where A, B are real numbers with $A^2 + B^2 = 1$. In this way we obtain, for each real α, and each trigonometric polynomial $T_n \in \mathcal{T}_n$ the inequality

$$(1.3) \qquad \left|\left|T_n\cos\alpha + \frac{1}{n}T_n'\sin\alpha\right|\right|_p \leq ||T_n||_p, \quad 0 \leq p \leq \infty.$$