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ATOMIC CHARACTERIZATIONS OF MODULATION SPACES THROUGH GABOR-TYPE REPRESENTATIONS

HANS G. FEICHTINGER

ABSTRACT. Given $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$ the modulation space $M^s_{p,q}(\mathbf{R}^m)$ can be described as follows, using the Gauss-function $g_0, g_0(x) := \exp(-x^2)$

$$M^s_{p,q}(\mathbf{R}^m) := \Big\{ \sigma |\ \sigma \in \mathcal{S}',\ g_0 * \sigma \in L^p(\mathbf{R}^m) \text{ and}$$
$$\|\sigma | M^s_{p,q}\| := \Big[\int_{\mathbf{R}^m} \|M_t g_0 * \sigma\|_p^q (1 + |t|)^{sq} \Big]^{1/q} < \infty \Big\}$$

(Writing M_t, $M_t f(x) := \exp(ix \cdot t) f(x), t, x \in \mathbf{R}^m$) for the modulation operator. Among these spaces one has the classical potential spaces $\mathcal{L}^2_s(\mathbf{R}^m) = M^s_{2,2}(\mathbf{R}^m)$ and the remarkable Segal algebra $S_0(\mathbf{R}^m) = M^0_{1,1}(\mathbf{R}^m)$. It is the aim of this paper to show that for these spaces an atomic characterization similar to known characterization of Besov spaces can be given (with dilation being replaced by modulation). Our main theorem is the following: Given $s \in \mathbf{R}$ and some $g_0 \neq 0$, $g_0 \in M^{|s|}_{1,1}(\mathbf{R}^m)$ (e.g., $g \in \mathcal{S}(\mathbf{R}^m)$ or $g \in L^1$ with compactly supported Fourier transform) one has:

THEOREM . *There exist $\alpha_0 > 0$ and $\beta_0 > 0$ such that, for $\alpha \leq \alpha_0$ and $\beta \leq \beta_0$, there exists $C = C(\alpha, \beta) > 0$ with the following property: $f \in M^s_{p,q}(\mathbf{R}^m)$ if and only if $f = \sum_{n,k} a_{n,k} M_{\beta n} L_{\alpha k} g_0$, for some double sequence of coefficients satisfying*

$$\Big[\sum_n \Big(\sum_k |a_{n,k}|^p \Big)^{q/p} (1 + |n|)^{sq} \Big]^{1/q} \leq C \|f | M^s_{p,q}(\mathbf{R}^m)\|.$$

The convergence is in the sense of tempered distributions, and in the norm sense for $p, q < \infty$.

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