ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 4, Fall 1988

TWO CONCRETE NEW CONSTRUCTIONS OF THE REAL NUMBERS

ARNOLD KNOPFMACHER AND JOHN KNOPFMACHER

ABSTRACT. Two new methods are put forward for constructing the complete ordered field of real numbers out of the ordered field of rational numbers. The methods are motivated by some known theorems on so-called Engel and Sylvester series. Amongst advantages of the methods are the facts that they do not require an arbitrary choice of "base", or any equivalence classes or similar constructs.

Introduction. By old theorems of Lambert (1770) and Engel (1913) (see Perron [**2**]), every real number A has a unique representation as the sum of a series

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \dots a_n} + \cdots = (a_0, a_1, a_2, \dots),$$

say, where the a_i are integers such that $a_{i+1} \geq a_i \geq 2$ for $i \geq 1$. Further, A is rational if and only if $a_{i+1} = a_i$ for all sufficiently large i.

An analogous representation (see [**2**]) of Lambert (1770) and Sylvester (1880) states that every real

$$A = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \cdots = ((a_0, a_1, a_2, \dots)),$$

say, where the a_i are integers defined uniquely by A, such that $a_1 \geq 2$ and $a_{i+1} \geq a_i(a_i - 1) + 1$ for $i \geq 1$. Further, A is rational if and only if $a_{i+1} = a_i(a_i - 1) + 1$ for all sufficiently large i.

In certain ways, these representations may be compared with that by "simple" continued fractions, and are even simpler than the latter. The main purpose of this note is to justify this remark by deriving some elementary further properties of the Engel-Lambert and Sylvester-Lambert representations, and (with these and the above-mentioned

Received by the editors on April 14, 1986.
AMS (1980) *Classification Number*. 10A30.