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A COEFFICIENT ESTIMATE FOR NONVANISHING H^p FUNCTIONS

JOHNNY E. BROWN AND JANICE B. WALKER

ABSTRACT. The Krzyz conjecture asserts that if $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ is a nonvanishing analytic function with $|f| \leq 1$ in $|z| < 1$, then $|a_n| \leq 2/e$ $(n = 1, 2, \ldots)$. Hummel, Scheinberg and Zalcman more generally conjectured that $|a_n| \leq (2/e)^{1/2}$ for all nonvanishing $f \in H^p$ with $||f||_p \leq 1 (1/p + 1/q = 1, 1 < p < \infty)$. We prove the latter conjecture for $n = 2$ and $n = 3$ for a natural subclass of nonvanishing H^p functions. We also point out a relationship between the two conjectures for this subclass. Our main tool in this investigation is the Pontryagin Maximum Principle.

1. Introduction. Let B_p denote the set of all nonvanishing H^p functions $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with $||f||_p \leq 1$. Hummel, Scheinberg and Zalcman [**4**] conjectured that

$$\sup_{B_p} |a_n| = \left(\frac{2}{e}\right)^{1/q}, \text{ for all } n \geq 1, \tag{1}$$

where $1 < p < \infty$ and $1/p + 1/q = 1$. If true, the bound is attained by

$$H_n(z) = \left(\frac{(1+z^n)^2}{2}\right)^{\frac{1}{p}} \left(\exp\left(\frac{z^n - 1}{z^n + 1}\right)\right)^{\frac{1}{q}}$$

and its rotations $e^{i\nu} H_n(e^{i\mu} z)$, where $\nu\mu \in \mathbf{R}$. To date, the only evidence supporting (1) is given in [**1**] where the conjecture was verified for $n = 1$ and for arbitrary $n \geq 2$ provided $a_m = 0$ for all $1 \leq m < (n+1)/2$. In this paper we prove the conjecture for $n = 2$ and $n = 3$ for a certain natural subclass of nonvanishing H^p functions which we now describe.

It is well-known (see [**2**] for example) that if f is a nonvanishing H^p function then

$$f(z) = e^{i\lambda} \Omega(z) I(z), \tag{2}$$

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