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LINEAR COMBINATIONS OF HYPONORMAL OPERATORS

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1. Introduction. It is well known that if N_1 and N_2 are two normal operators such that their linear span consists of normal operators, then N_1 and N_2 commute [**8**]. This paper addresses the question whether this is true for hyponormal and subnormal operators. The answer is no. Two noncommuting hyponormal operators are given in this note such that their linear span consists entirely of hyponormal operators. What is true (Proposition 2.3)is that if A and B are two hyponormal operators and if $AB^* = B^*A$, then the linear span of A and B consists of hyponormal operators and both AB and BA are hyponormal.

The linear span of two normal operators consists entirely of normal operators if and only if the operators commute. There are, however, examples of subnormal operators A and B such that $AB = BA$ but neither $A + B$ nor AB is hyponormal [**7**] (also see pages 23-24 of [**5**]). Since half of this result for normal operators fails to generalize to the hyponormal case it is not too surprising that the other half also fails to generalize. The counterexample demonstrating this (Example 2.4) arises by constructing hyponormal operators A and B such that $AB^* = B^*A$, but $AB \neq BA$. In light of the result mentioned above, this shows that $\text{span}\{A, B\} \equiv \{aA + bB : a, b \in \mathbf{C}\}$ consists of hyponormal operators even though A and B do not commute.

This leads to a consideration of the question, "If A and B are hyponormal operators and $AB^* = B^*A$, when does $AB = BA$"? That is, when is the converse of Fuglede's Theorem valid for hyponormal operators? It is well known that Fuglede's Theorem does not extend to hyponormal operators or even subnormal operators. Moreover, as mentioned above, the possible generalization of the converse of

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