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TWO THEOREMS ON INVERSE INTERPOLATION

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ABSTRACT. The usual task of interpolation theory is, given a function f, or some of its properties, to find out what properties the set $\mathcal{L}(f)$, of all Lagrange interpolants of f, must have. What we mean by *inverse interpolation* is to reverse this body of problems. Namely, given the set $\mathcal{C}(f)$ or some of its properties, to recover f or some of its properties. We stress that $\mathcal{L}(f)$ is considered as an unstructured set of polynomials.

Our first result asserts that if f is analytic on the unit interval, then f is completely determined by the set $\mathcal{L}(f)$. Our second result constructs a large class of infinitely differentiable functions f on the unit interval, such that $\mathcal{L}\mathcal{L}(f) = \mathcal{P}$, the set of all polynomials. In other words, every polynomial in the world is a Lagrange interpolant of a Lagrange interpolant of f. Thus, such an f is in no wise recoverable from $\mathcal{L}(\mathcal{L}(f))$. So on the one hand, $\mathcal{L}(f)$ determines f if f is analytic on $[0,1]$, while on the other hand, $\mathcal{L}(\mathcal{L}(f))$ does not determine f if f is only assumed C^∞ on $[0,1]$. There is clearly a gap in our knowledge here that should be closed-see the problems at the end of the paper. In several further papers we are now preparing, we pursue such related questions as, "if we assume a uniform bound on all the Lagrange interpolants of f, what does this tell us about f"?

If f is a real-valued function on a set S, we say that a polynomial p, say of degree n, is a Lagrange interpolant of f, if there exist $n+1$ distinct numbers $x_0, x_1, \ldots x_n$ in S such that $f(x_i) = p(x_i)$ for $i = 0, 1, \ldots, n$. Of course there may be other points x where $p(x) = f(x)$. Then p must be given by the usual Lagrange interpolation formula

$$p(x) = \sum_{k=0}^{n} f(x_k) l_k(x),$$

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