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DUAL MODULES AND GROUP ACTIONS ON EXTRA-SPECIAL GROUPS

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1. Introduction. When constructing examples or counter-examples in the theory of solvable groups, it is often the case that what is needed is some group which acts in an interesting way on an extra-special p-group. Specifically, what we have in mind is the following.

Let G be a finite group and let V be an irreducible FG-module where $F = GF(p)$. It is easy to construct an extra-special p-group E acted on by G such that $E = AB$ where A and B are G-invariant elementary abelian normal subgroups with $A \cap B = Z = \mathbf{Z}(E)$. This can be done so that A/Z is FG-isomorphic to V and B/Z is FG-isomorphic to the "dual" or contragredient FG-module V^*. Furthermore, G acts trivially on Z.

Now comes the more subtle part. Suppose $G \triangleleft \Gamma$ where $|\Gamma : G| = 2$ and where the conjugation action of Γ on the set of isomorphism classes of FG-modules interchanges the classes of V and V^*. (We allow the possibility that $V \simeq V^*$ and this isomorphism class is Γ-invariant.) The question is whether or not the action of G on E can be extended to a Γ-action in which the elements of $\Gamma - G$ interchange A and B.

The answer is "yes".

THEOREM A. *Let $G \triangleleft \Gamma$ with $|\Gamma : G| = 2$ and let V be an irreducible FG-module where $F = GF(p)$. Assume that V is conjugate to V^* in Γ. Then Γ acts on an extra-special p-group E and the following hold.*

a) *$E = AB$ where $A, B \triangleleft E$ are elementary abelian and $A \cap B = \mathbf{Z}(E)$.*

b) *G centralizes $Z = \mathbf{Z}(E)$ and acts on A/Z and B/Z as it does on V and V^* respectively.*

c) *The elements of $\Gamma - G$ interchange A and B and either all of them centralize or else all of them invert Z. Furthermore, the choice of the*

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