ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 18, Number 2, Spring 1988

RESOLVING SINGULAR NONLINEAR EQUATIONS

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ABSTRACT. This paper concerns the solutions of operator equations $G(z,\lambda) = 0$ having solutions (z_0,λ_0) for which $G'(z_0,\lambda_0)$ is not a surjection. More precisely, suppose $\lambda \in R^q, q \geq 0$ and $\dim N(G'(z_0,\lambda_0)) = m+q > 0$, where $N(\cdot)$ denotes the kernel. Several different kinds of singular problems can be treated in a unified way. Examples are parameter dependent problems with $q > 0$ and $m > 1$ and operator equations with $m > 0, q = 0$. In the latter case the corresponding discrete analogues also have some corresponding singularities which usually lead to the breakdown of numerical solution techniques. The former case includes multiple bifurcations for multi-parameter problems. The main results involve the construction of an inflated map $H(z,\lambda,\dots)$ (where $\dots$ denotes additional augmented variables). The map H has an invertible derivative at $(z_0,\lambda_0,\dots)$ and a component $F(z,\lambda,c)$ such that $F(z,\lambda,0) = G(z,\lambda)$. This H may be used to define quadratically convergent Newton methods. Several examples of finite dimensional equations and operator equations are studied. In practical applications m is often not known a priori. Some ways of determining $m + q$ are described.

1. Introduction. In this paper we consider operators

$$G : \mathbf{E} := \mathbf{E}_0 \times \mathbf{R}^q \to \hat{\mathbf{E}}, \quad (\mathbf{E}_0, \hat{\mathbf{E}} \text{ are Banach spaces, } q \geq 0), \tag{1.1}$$

in the neighborhood of a zero point (z_0,λ_0) of G having a nontrivial null space of the Frechet derivative $G' = (G_z, G_\lambda)$. Here G_z, G_λ denote the partial derivatives with respect to z and λ, respectively. Letting $\mathbf{N}(L)$ denote the kernel of a linear operator L, we have

$$G(z_0,\lambda_0) = 0, \quad \dim \mathbf{N}(G'(z_0,\lambda_0)) = m + q > 0. \tag{1.2}$$

In case $q > 0$ we may have the usual bifurcation problem in continuation methods, which is discussed in the literature primarily for $m = 1$. For $q = 0$ and $m > 0$, the usual discretization methods for the computation of an isolated solution usually fail. Hence, modifications are necessary.

Received by the editors on April 7, 1986 and in revised form on September 15, 1986.