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# THE INTERMEDIATE VALUE THEOREM: PREIMAGES OF COMPACT SETS UNDER UNIFORMLY CONTINUOUS FUNCTIONS

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ABSTRACT. A strong constructive form of the intermediate value theorem is established. Let $f$ be a uniformly continuous map from a connected, locally connected, compact metric space $X$ to the real numbers $\mathbf{R}$ with $\alpha < \beta$ in the range of $f$. Except for countably many real numbers $r$, if $\alpha < r < \beta$, then the set $f^{-1}(r)$ is nonempty and compact. An application is a constructive proof of the Schoenflies theorem that the interior of a Jordan curve in the plane is homeomorphic to a disk.

**0. Introduction.** The intermediate value theorem is often cited as a theorem from classical mathematics which is not constructively valid. The standard Brouwerian counter example is as follows.

EXAMPLE 0.1. Let $a$ be a real number and define the function $f$ by $f(0) = -1, f(1/3) = 0 = f(2/3), f(1) = 1$, and linear inbetween. Solving the equation $f(x) = a$ for $-1 \leq a \leq 1$ is equivalent to determining whether $a \geq 0$ or $a \leq 0$.

Bishop [**2**] proves the following constructive form of the intermediate value theorem:

*Let $f : [0,1] \to \mathbf{R}$ be uniformly continuous and let $\alpha < \beta$ be in the range of $f$. If $\alpha < r < \beta$ and $\varepsilon > 0$, then $f^{-1}(r-\varepsilon, r+\varepsilon)$ is nonempty.*

The following stronger constructive version of the intermediate value theorem for $X = [0,1]$ is Problem 15 in [**2**, page 110].

(A) *Let $f : X \to \mathbf{R}$ be uniformly continuous with $\alpha < \beta$ in $fX = \{f(x) : x \in X\}$. For all but countably many real numbers $r$, if $\alpha < r < \beta$, then the set $f^{-1}(r)$ is nonempty and compact.*

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