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ON THE ISOMORPHISM PROBLEM FOR GROUP RINGS AND COMPLETED AUGMENTATION IDEALS

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1. Introduction. Let G be a group, $\Delta_{\mathbf{Z}}G$ its integral and ΔG its modular augmentation ideal, i.e., over the field $\mathbf{F}_p$ of p elements. In this note, we consider the integral isomorphism problem–whether $\Delta_{\mathbf{Z}}G = \Delta_{\mathbf{Z}}H$ implies $G \xrightarrow{\sim} H$– for certain finite p–groups and emphasize the aspect of how much of our methods carry over to the modular case.

Having finite p–groups at our disposal, the most obvious approach to attack the isomorphism problem is to try some kind of induction. To put this idea to work, two ingredients turn out to be essential: One has to be able to lift automorphisms of $\Delta_{\mathbf{Z}}\hat{G}$, resp. $\Delta\hat{G}, \hat{G}$ a homomorphic image of G, to automorphisms of the augmentation ideal of a free group F with $F \twoheadrightarrow G$, and the lifting has to leave certain ideals invariant. Although it is not possible to solve the first problem in general, since the group ring of a free group does not contain enough units, one can do it for the completed group rings: Lemma 3.1 gives a solution, which can be easily generalized to other rings of coefficients.

However, to guarantee that the lifting is again an automorphism, we have to impose one further condition on the automorphisms under consideration: They have to induce the identity on $\Delta\hat{G}/\Delta^2\hat{G}$. Although this seems to be a severe restriction on the first glance, it is always satisfied for the start of the induction (and looks rather natural for these cases). The isomorphism $A \xrightarrow{\sim} \Delta_{\mathbf{Z}}A/\Delta_{\mathbf{Z}}^2 A$ for an abelian group A gives, in case $\Delta_{\mathbf{Z}}A = \Delta_{\mathbf{Z}}B$, an automorphism of $\mathbf{Z}A$ sending A onto B with the above property, and hence, even the Whitcomb isomorphism for metabelian torsion groups has it. Furthermore, the lifting, too, has this property (see (3.1)).

All these considerations lead to the following concept: A group G is $\mathbf{F}_p$–strongly characterized by its integral group ring, if $\Delta_{\mathbf{Z}}G = \Delta_{\mathbf{Z}}H$ implies the existence of an isomorphism $G \xrightarrow{\sim} H$, whose extension to an automorphism of $\Delta_{\mathbf{Z}}G$ induces the identity on $\Delta G/\Delta^2 G$. Although

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