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SOLUTIONS OF EQUATIONS OVER ω–NILPOTENT GROUPS

S.M. GERSTEN

While studying the general question of solving equations in groups, I came across the curious fact the Newton's algorithm applies in a variety of noncommutative situations. In particular, if G is an ω–nilpotent group and $w \in G * \langle t \rangle$ is an equation with exponent sum $e_t(w) = \pm 1$, then $w(t) = 1$ has a unique solution in $U_1(\mathbf{Z}[G]\hat{})$. Here $\mathbf{Z}[G]\hat{}$ is the completion of the integral group ring $\mathbf{Z}[G]$ in the I_G–adic topology, I_G the augmentation ideal, and U_1 refers to units $\equiv 1 \bmod I_G\hat{}$.

More generally, let A be a ring equipped with a descending filtration $\{\sigma_n\}$ of two sided ideals (so $\sigma_0 = A, \sigma_i \cdot \sigma_j \subseteq \sigma_{i+j}$). Assume A is complete–i.e. the canonical map $A \to \varprojlim_n A/\sigma_n$ is an isomorphism. Let $U_1(A)$ denote the group of units of A congruent to 1 modulo σ_1. Let $w \in U_1(A) * \langle t \rangle$ be such that $e_t(w)$ is a unit in A. Then the equation $w(t) = 1$ has a unique solution in $U_1(A)$.

Applied to $\mathbf{Q}G$, where G is *fg* torsion free nilpotent, this implies the classical result that $U_1(\mathbf{Q}G\hat{})$ contains the Mal'cev completion [**2**] of G.

A quick homological proof is offered of the Kervaire conjecture for ω–nilpotent groups. I am aware there are other proofs based on residual properties. With that out of the way, the proper topic of this paper, uniqueness of solutions, can begin.

This work was done while I was on sabbatical leave from the University of Utah.

1. Existence of solutions. Let G be a group, $G\langle t \rangle$ the free product of G with an infinite cycle $\langle t \rangle$, and $w(t) \in G\langle t \rangle$. Let $e_t(w)$ be the exponent sum of t in $w(t)$. The Kervaire conjecture is that G injects in $G\langle t \rangle / N$, where N is the normal closure of $w(t)$ in $G\langle t \rangle$, provided $e_t(w) = \pm 1$. This is equivalent to the existence of a group G_1 containing G as a subgroup and an element $x \in G_1$ such that $w(x) = 1$.

Let $\Gamma_n(G)$ denote the lower central series of a group G; thus $\Gamma_0(G) = G$ and $\Gamma_{n+1}(G) = (G, \Gamma_n(G))$. Similarly, if p is a prime number, let $\Gamma_{n,p}(G)$ be the p–lower central series; thus $\Gamma_{o,p}(G) = G$ and $\Gamma_{n+1,p}(G)$

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