ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 4, Fall 1987

SPHERES WITH CONTINUOUS TANGENT PLANES

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1. Introduction. Burgess [2] soved Problem 12 in *The Scottish Book* by exhibiting a wild 2–sphere in E^3 having a continuous family of tangent planes. A 2–sphere Σ in E^3 is said to be *wild* if no space homeomorphism takes Σ onto the sphere S defined by $\{(x,y,z)|x^2+y^2+z^2=1\}$. Spheres that are not wild are called *flat* or *tame.* The definition of a plane being tangent to a surface Σ in Euclidean 3–space E^3 comes from Problem 156 of *The Scottish Book.* A plane $T(q)$ is *tangent* to Σ at a point q of Σ if, for each positive number ϵ, there exists a round ball B centered at q such that the measure of the angle between $T(q)$ and every straight line $L(q,x)$ determined by q and a point x of $\Sigma \cap B \backslash \{q\}$ is less than ϵ. A surface may have infinitely many tangent planes at a single point q as ones sees by examining the surface obtained by rotating the graph of $|x|^{1/2}+|z|^{1/2}=1$ about the z–axis and letting $q=(0,0,1)$, see Figure 1. A 2–sphere Σ is said to have *continuous tangent planes* over a subset K of Σ if, for each q in K, there is a unique tangent plane $T(q)$ to Σ at q such that $\{T(q_i)\}$ converges to $T(q)$ whenever $\{q_i\}$ is a sequence of points of K converging to q. When we say Σ has a continuous family of tangent planes we mean to take K equal to Σ.

The wildness of the spheres described by Burgess [2] occurs at points of the 2–sphere Σ that belong to its rim. The *rim* R of Σ is the set of all points q of Σ where the normal to some tangent plane to Σ at q fails to pierce Σ at q. In [2] the rim of Σ is a simple closed curve containing the single wild point of Σ. The original motivation for this paper came from a desire to better understand the rim of Σ and its relation to the wild set. A point q of a 2–sphere Σ in E^3 is said to belong to the *wild set* W of Σ if there is no 2–cell K in Σ such that q lies in $\operatorname{Int} K$ and K lies on a tame 2–sphere in E^3. Example 4.2 describes a 2–sphere Σ in E^3 with a continuous family of tangent planes, a 1–dimemsional wild set, and a rim that is the union of a countable sequence of disjoint simple closed curves.

Received by the editors on January 13, 1983 and in revised form on November 11, 1983.