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QUASI–COHERENT MODULES ON QUASI–AFFINE SCHEMES

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ABSTRACT. It is shown that a quasi–coherent sheaf of modules on a quasi–compact open subset of an affine scheme can be realized as an object in a subcategory of a module category. In particular, the modules of sections is canonically isomorphic to a (torsion theoretic) localized module. This generalizes the noetherian case of P.–J. Cahen. A few simple examples exploit this relationship.

1. Introduction. If A is a noetherian ring and U is an open subset of $X = \operatorname{Spec} A$, then P.–J. Cahen [1, Theorem 6.1] has shown, by torsion theoretic methods, that for any A–-module M, the module of sections $r(u, \tilde{M})$ of the quasi–coherent Q_x–module $\tilde{M}$ is the module of quotients $Q_U(M) = \lim \rightarrow \operatorname{Hom}(I, M)$ where the direct limit is taken over the set $\phi_u = \{I \subseteq A | \forall p \in U,\ I \not\subseteq p\}$. Our aim is to generalize this result to an arbitrary (commutative) ring in the case U is a quasi–compact open subset of $\operatorname{Spec} A$.

We show that for any such U : 1) every quasi–coherent Q_U–module F is the restriction to U of some quasi–coherent $\mathcal{Q}_X$–module $\tilde{M}$; 2) if $\tilde{M}$ is any extension of F, the module of sections $\Gamma(U, F) = \Gamma(U, \tilde{M})$ is just the module of quotients $Q_U(M) = \lim \rightarrow \operatorname{Hom}(I, \overline{M})$ where $\overline{M} = M/T_u(M)$ and $T_U(M) = \{x \in M | (0 : x)\varepsilon_U^{\phi}\}$ is the torsion submodule of M with respect to the torsion class T_U; and 3) the category of quasi–coherent $\mathcal{Q}_U$–modules is equivalent to the category $(A, T_U) - \operatorname{mod}$. Here $(A, T_U) - \operatorname{mod}$ is the full subcategory $\{M \in A - \operatorname{mod} | \phi_M : M = \operatorname{Hom}(A, M) \rightarrow Q_U(M)$ is an isomorphism $\}$. As a corollary, torsion theoretic methods in $(A, T_U) - \operatorname{mod}$ yield interesting proofs of generalizations of standard theorems in algebraic geometry as well as new theorems in this class of $\mathcal{Q}_U$–modules. We give an example of the latter by characterizing the injective objects in the category of quasi–coherent $\mathcal{Q}_U$–modules.

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