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COMPACT WEIGHTED COMPOSITION OPERATORS ON SOBOLEV RELATED SPACES

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ABSTRACT. If m is a positive integer and $1 \leq p \leq \infty$, let $W_{m,p}$ denote the set of functions f on the unit interval $[0,1]$ for which $f, f', \dots, f^{(m-1)}$ are absolutely continuous and $f^{(m)} \in L^p$. With $\|f\|_{W_{m,p}} = \left(\sum_{s=0}^{m} \|f^{(s)}\|_p^p\right)^{1/p}$, $1 \leq p < \infty$, $W_{m,p}$ is a Banach space. We show that if $u \in W_{m,\infty}, \varphi : [0,1] \to [0,1], \varphi \in W_{m,\infty} \cap C^1$, and there exists a positive integer N for which $\varphi^{-1}([a,b])$ can be expressed as a union of N intervals for all $a, b \in [0,1]$, then the weighted composition operator $uC_\varphi : f(x) \to u(x)f(\varphi(x))$ is a bounded linear operator on $W_{m,p}$ which is compact if and only if $u\varphi' = 0$. Further, if uC_φ is compact on $W_{m,p}$, then the spectrum $\sigma(uC_\varphi) = \{\lambda | \lambda^n = u(c) \dots u(\varphi_{n-1}(c))$ for some positive integer n and some fixed point c of φ of order $n\} \cup \{0\}$.

If m is a positive integer and $1 \leq p \leq \infty$ let $W_{m,p}$ denote the set of functions f on $[0,1]$ for which f and the derivatives $f', f'', \dots, f^{(m-1)}$ lie in AC, the space of absolutely continuous functions on $[0,1]$, and $f^{(m)} \in L^p(0,1) \equiv L^p$. For $1 \leq p < \infty, W_{m,p}$ is a Banach space under the norm $\|f\|_{W_{m,p}} = \left(\sum_{s=0}^{m} \|f^{(s)}\|_p^p\right)^{1/p}$. These spaces are closely related to Sobolev spaces on $[0,1]$ (see [**1,2,3**]). A weighted composition operator on $W_{m,p}$ is a map from $W_{m,p}$ to itself of the form $f(x) \to u(x)f(\varphi(x))$, where $u : [0,1] \to \mathbf{C}$ and $\varphi : [0,1] \to [0,1]$. We denote such a map by uC_φ.

In [**1**] Antonevich considered weighted composition operators on $W_{m,p}$, where $u, \varphi \in C^m[0,1]$ and φ is a bijection of $[0,1]$ onto itself and determined their spectra. In this note we study other weighted composition operators on $W_{m,p}$ and characterize those operators which are compact. We show that if $u \in W_{m,\infty}, \varphi \in W_{m,\infty} \cap C^1$ and if $uC_\varphi : W_{m,p} \to W_{m,p}$, then uC_φ is compact if and only if $u\varphi' = 0$. Further, if we let φ_n denote the n^{th} iterate of φ and $\sigma(uC_\varphi)$ the spectrum of uC_φ, then if uC_φ is compact on $W_{m,p}$, we have that $\sigma(uC_\varphi)\backslash\{0\} = \{\lambda | \lambda^n = u(c) \dots u(\varphi_{n-1}(c))$ for some positive integer

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