ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

SOME JESSEN–BECKENBACH INEQUALITIES

JOSIP E. PEČARIĆ AND PAUL R. BEESACK

1. Introduction. In 1966 E.F. Beckenback [**1**] (see also [**4**, p.52] or [**5**, p.81] proved the following generalization of Hölder's inequality:

Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ be two n–tuples of positive real numbers, and p, q be real numbers such that $p^{-1} + q^{-1} = 1 (p > 1)$. If $0 < m < n$, then

$$(1) \qquad \Big(\sum_1^n a_i^p\Big)^{1/p}\Big(\sum_1^n a_i b_i\Big)^{-1} \geq \Big(\sum_1^n \tilde{a}_i^p\Big)^{1/p}\Big(\sum_1^n \tilde{a}_i b_i\Big)^{-1},$$

where

$$\tilde{a}_i = a_i (1 \leq i \leq m), \quad \tilde{a}_i = \Big\{ b_i \sum_{j=1}^m a_j^p \Big/ \sum_{j=1}^m a_j b_j \Big\}^{q/p} \quad (m+1 \leq i \leq n).$$

Equality holds in (1) *if and only if $\tilde{a}_i \equiv a_i$. The inequality in* (1) *is reversed if $p < 1, p \neq 0$. For $m = 1$,* (1) *reduces to Hölder's inequality.*

In this paper we shall give some generalizations of this result with $\sum$ replaced by an isotonic linear functional. See especially Corollary 3, and Remark 4, below.

2. Main results. Let E be a nonempty set, let $\mathcal{A}$ be an algebra of subsets of E, and let L be a linear class of real–valued functions $g : E \to \mathbf{R}$ having the properties

L1: *$f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in \mathbf{R}$;*
L2: *$1 \in L$, that is if $f(t) = 1$ for $t \in E$, then $f \in L$;*
L3: *$f \in L, E_1 \in \mathcal{A} \Rightarrow fC_{E_1} \in L$,*

where C_{E_1} is the characteristic function of $E_1 (C_{E_1}(t) = 1$ for $t \in E_1$, or 0 if $t \in E \backslash E_1)$. It follows from L2, L3 that $C_{E_1} \in L$ for all $E_1 \in \mathcal{A}$. Also note that L contains all constant functions by L1, L2.

We also consider isotonic linear functionals $A : L \to \mathbf{R}$. That is, we

Received by the editors on November 26, 1985.