ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

OSCILLATORY AND ASYMPTOTIC BEHAVIOR IN CERTAIN THIRD ORDER DIFFERENCE EQUATIONS

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1. Introduction. In this paper, the difference equation

$$\Delta^3 U_n - P_n U_{n+2} = 0, \tag{E^-}$$

where Δ denotes the differencing operation $\Delta U_n = U_{n+1} - U_n$, will be studied subject to the condition $P_n > 0$ for each integer $n \geq 1$. An example is given which shows that it is possible for (E^-) to have only nonoscillatory solutions. Our main result is a discrete analogue of Taylor [**15**, Theorem 6], and is concerned with a characterization of the existence of oscillatory solutions of (E^-) in terms of the behavior of nonoscillatory solutions. We also refer to the works of Hanan [**3**], Jones [**5,6**] and Lazer [**7**].

We will use primarily the terminology of Fort's Book [**1**] in our discussion. A real sequence $U = \{U_n\}$ which satisfies (E^-) for each $n \geq 1$ we term a solution of (E^-). Hereafter the term "solution" shall mean a "nontrivial solution." By the graph of a solution U we will mean the polygonal path connecting the points $(n, U_n), n \geq 1$. Any point where the graph of U intersects the real axis is called a node. A solution of (E^-) will be called oscillatory if it has arbitrarily large nodes; otherwise it is said to be nonoscillatory. Owing to the linearity of (E^-), we assume without loss of generality that all nonoscillatory solutions are eventually positive. Whenever (E^-) has an oscillatory solution we say that (E^-) is oscillatory. It is understood below that the variables n, m, N, M, i, j, k represent positive integers.

2. Preliminary Results. Our first result shows that initial values can be used to construct nonoscillatory solutions of (E^-). Since the proof is an easy argument, using the technique of mathematical induction, it will be omitted.

LEMMA 2.1. *If U is a solution of (E^-) satisfying*

$$U_m \geq 0, \Delta U_m \geq 0, \Delta^2 U_m > 0,$$

Received by the editors on August 9, 1984, and in revised form on June 18, 1985.