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ARITHMETIC PROGRESSIONS IN LACUNARY SETS

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ABSTRACT. We make some observations concerning the conjecture of Erdös that if the sum of the reciprocals of a set A of positive integers diverges, then A contains arbitrarily long arithmetic progressions. We show, for example, that one can assume without loss of generality that A is lacunary. We also show that several special cases of the conjecture are true.

1. Introduction. The now famous theorem of Szemerédi [7] is often stated:

(a) *If the density of a set A of natural numbers is positive, then A contains arbitrarily long arithmetic progressions.*

Let us call a set A of natural numbers k-good if A contains a k-term arithmetic progression. Call A ω-good if A is k-good for all $k \geq 1$. We define four density functions as follows: For a set A and natural numbers m, n, let $A[m, n]$ be the cardinality of the set $A \bigcap \{m, m+1, m+2, \ldots, n\}$. Then define

$$\underline{\delta}(A) = \liminf_{n} \frac{A[1, n]}{n},$$
$$\overline{\delta}(A) = \limsup_{n} \frac{A[1, n]}{n},$$
$$\underline{u}(A) = \lim_{n} \min_{m \geq 0} \frac{A[m+1, m+n]}{n} \text{ and}$$
$$\overline{u}(A) = \lim_{n} \max_{m \geq 0} \frac{A[m+1, m+n]}{n}.$$

It can be seen that the limits in the definitions of $\underline{u}$ and $\overline{u}$ always exist. These four "asymptotic" set functions are called the lower and upper "ordinary" and the lower and upper "uniform" density of the set A respectively. They are related by

$$\underline{u}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{u}(A)$$

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