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DIRECT SUMS AND PRODUCTS OF ISOMORPHIC ABELIAN GROUPS

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Introduction. Suppose G is a reduced abelian group and I and J are infinite sets. When can the direct product G^I equal the direct sum $A^{(J)}$ for some subgroup A? If G is a torsion group, then G must be torsion by Corollary 2.4 in [**3**] and the answer is easy to determine. In Theorem 1 we provide an answer for all cases where $|G|$ or $|I|$ is non-measurable. We then present, in Example 2, a group decomposition $G^I = A^{(J)}$ where G is reduced and unbounded. There is another unusual decomposition of G^I which occurs whenever $|I|$ is measurable and seems worth mentioning. We do this in Example 3.

In this paper all groups are abelian. By G^I and $G^{(I)}$ we mean the direct product and direct sum respectively of copies of G indexed by I. If I is a set, then $|I|$ is measurable if there is a $\{0,1\}$-valued countably additive function μ on $P(I)$, the power set of I such that $\mu(I) = 1$ and $\mu(\{i\}) = 0$ for each $i \in I$. The letter N denotes the set of natural numbers. Unexplained terminology may be found in [**2**].

THEOREM 1. *Let G be a reduced group and let I and J be infinite sets. If $|G|$ or $|I|$ is non-measurable, then $G^I = A^{(J)}$ for some subgroup A if and only if $G = B \oplus C$, where $B^I \cong T^{(J)}$ for some bounded subgroup T and $C^I \cong C^{(J)} \cong C^k$ for some positive integer k.*

PROOF. Sufficiency is clear so we assume $G^I = A^{(J)}$ and derive the stated conditions. Write $X = \prod_I G_i = \oplus_J A_j$ where $\phi_i : G_i \to G$ is an isomorphism for each i and $A_J \cong A$ for each j.

(A) Suppose $|G|$ is non-measurable. Let $f_j : X \to A_j$ be the obvious protection and let $(S, +, \cdot)$ be the Boolean ring on $S = P(I)$. Also let $K = \{s \in S$: there is an n_s in N such that $n_s f_j(\prod_s G_i) = 0$ for almost all $j\}$ and set $H = \langle \prod_s G_i : s \in K\rangle$. Clearly K is an ideal in S. Thus H consists of the elements in G with support in K. The crucial fact for our proof is that K is a γ-ideal in S (i.e., if $\{s_n : n \in N\}$ is an

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