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PROJECTIVE MAPPINGS ON DIFFERENTIABLE MANIFOLDS

NIRMALA PRAKASH

ABSTRACT. A mapping $f : M \to M'$ between two C^∞-manifolds is quasi-projective if it carries geodesics to geodesics, and if, in addition, it preserves the projective parameter, it is called projective. Such a mapping is known to relate the symmetric affine connections of M and M', and is characterised by a relation between the Schwartzian differential (for the parameters) and the Ricci curvatures of M and M'. We use these facts to establish the non-existence and existence of projective maps. For instance we show that f is not projective if there does not exist a solution to the non-linear non-homogeneous equation given by the Schwartzian differential and the Ricci tensor; it is projective if f is a diffeomorphism and the Schwartzian differential formed by the projective parameters is zero. We also use the collection of projective maps on M to define an action integral on it and show that the extremal of this action is a Levi-Civita connection. Finally, we prove that if energy-momentum tensors and sectional curvatures are suitably restricted then a quasi-projective (projective) mapping can be volume (distance) decreasing.

Given a C^∞-manifold M, consider the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. Let $\mathcal{I}(M)$ and $\mathcal{A}(M)$ denote the group of isometrics and affine mappings on M. Then the following inclusion relation is a classical fact:

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathrm{Diff}(M).$$

A diffeomorphism which (1) carries geodesics to geodesics and (2) preserves the projective parameter p (up to linear fractional transformations) is called projective. The set of all these diffeomorphisms is a group denoted $\mathcal{P}(M)$. It is easy to see that

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathcal{P}(M) \subset \mathrm{Diff}(M).$$

The class of diffeomorphisms which does not necessarily satisfy (2) also forms a group denoted $\tilde{\mathcal{P}}(M)$, evidently $\tilde{\mathcal{P}}(M) \supset \mathcal{P}(M)$. Thus we have

$$\mathcal{I}(M) \subset \mathcal{A}(M) \subset \mathcal{P}(M) \subset \tilde{\mathcal{P}}(M) \subset \mathrm{Diff}(M).$$

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