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A NOTE ON PURE RESOLUTIONS OF POINTS IN GENERIC POSITION IN $\mathbf{P}_k^n$

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Preliminaries. Let $P_1, \dots P_s \in \mathbf{P}_k^n$ denote s distinct points in projective n-space $\mathbf{P}_k^n$. Throughout this paper, k will denote an algebraically closed field whose characteristic is arbitrary (except for a couple of examples near the end). Let $S = k[X_0, \dots, X_n]$ denote the polynomial ring in $n+1$ variables $X_0, \dots, X_n$ over k. If I denotes the ideal of $P_1, \dots, P_s$ in S, then I is a perfect, unmixed, radical ideal of grade n. Let $R = S/I$, the coordinate ring of $P_1, \dots, P_s$. R is a standard graded k-algebra, Cohen-Macaulay of dimension one and has projective dimension n as an S-module. Thus, R has a minimal, free resolution Γ of the form:

$$\begin{aligned} \Gamma : 0 \to \oplus_{i=1}^{\beta_n} S(-d_i^{(n)}) \xrightarrow{\phi_n} \cdots \to \oplus_{i=1}^{\beta_2} S(-d_i^{(2)}) \xrightarrow{\phi_2} \\ \to \oplus_{i=1}^{\beta_1} S(-d_i^{(1)}) \xrightarrow{\phi_1} S \to R \to 0. \end{aligned} \tag{1}$$

In (1), $\beta_1, \dots, \beta_n$ are the nontrivial betti numbers of R. Each ϕ_1 is a homogeneous, S-module homomorphism of degree zero. Consequently, ϕ_i can be represented by a $\beta_i \times \beta_{i-1}$ matrix $(\alpha_{pq}^{(i)})$ where $\alpha_{pq}^{(i)}$ is a homogeneous form in S of degree $\partial(\alpha_{pq}^{(i)}) = d_p^{(i)} - d_q^{(i-1)}$. Γ being minimal means every $\alpha_{p,q}^{(i)} \in (X_0, \dots, X_n)$. The $d_i^{(j)}$ in (1) are called the twisting numbers of R and, along with $\beta_1, \dots, \beta_n$, are unique.

We say $P_1, \dots, P_s$ have a pure resolution of type $(d_1, \dots, d_n)$ if, in the minimal resolution (1) of R, we have for all $j = 1, \dots, n$ and for all $i = 1, \dots, \beta_j, d_i^{(j)} = d_j$. Thus, $P_1, \dots, P_s$ have a pure resolution of type $(d_1, \dots, d_n)$ with betti numbers $\beta_1, \dots, \beta_n$ if and only if the minimal, free resolution Γ of R has the simple form:

$$\Gamma : 0 \to S(-d_n)^{\beta_n} \to \cdots \to S(-d_2)^{\beta_2} \to S(-d_1)^{\beta_1} \to S \to R \to 0. \tag{2}$$

We note that the minimality of Γ implies $0 < d_1 < d_2 < \cdots < d_n$ in (2).

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