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THE STRONG FORM OF AHLFORS' LEMMA

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1. Introduction. Ahlfors [**1**] established an extension of Schwarz' lemma which plays an important role in geometric function theory. Ahlfors' lemma is valid for any hyperbolic Riemann surface; let us recall the result for the open unit disk D. If $\rho(z)|dz|$ is an ultrahyperbolic metric on D, then

$$\rho(z) \leq \lambda_D(z) = \frac{2}{1-|z|^2}. \tag{1}$$

Here $\lambda_D(z)|dz|$ is the hyperbolic metric on D normalized to have curvature -1. (In some references the curvature is taken to be -4; we will translate all such results to the context of curvature -1 without further comment.) The proof of (1) is astonishingly elementary; it relies on the fact that the Laplacian of a real-valued function is nonpositive at any point where the function has a relative maximum. Ahlfors did not show that equality in (1) at a single point implied $\rho = \lambda_D$ which would be the analog of the equality statement in Schwarz's lemma. Heins [**2**] introduced the class of SK metrics, which includes ultrahyperbolic metrics, and verified that (1) remains valid for SK metrics. In addition, he showed that equality at a single point implied $\rho = \lambda_D$. However, his proof of the equality statement is not as elementary as the proof of Ahlfors' lemma since it relies on an integral representation for a solution of the nonlinear partial differential equation $\Delta u = \exp(2u)$. In this paper we shall present an elementary proof of the equality statement for Ahlfors' lemma for SK metrics. Our proof is in the spirit of Ahlfors' derivation of (1) and is a modification of a method introduced by Hopf [**3**] for linear partial differential equations. A related proof was given by Jϕrgensen [**4**] in the special case of metrics with constant curvature -1.

2. Maximum principle. We prove a strong maximum principle for upper semicontinuous functions which satisfy the differential inequality

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