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ON WEAK RECURRENT POINTS OF ULTIMATELY NONEXPANSIVE MAPPINGS

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ABSTRACT. Conditions are investigated which imply that every ultimately nonexpansive mapping has a weak recurrent point. Examples are presented which show that better results can not be obtained.

1. Introduction. Throughout this paper X will represent a real Banach space, H a real Hilbert space, and K a weakly compact (usually convex) subset of either X or H. The functions f and F will be ultimately nonexpansive self mappings (see below) of K. In §2 f is arbitrary, while f and F are specific functions in §3 and §4.

DEFINITIONS. A fixed point of a self mapping f of a set K is a point x such that $f(x) = x$.

A function $f : K \to K \subseteq X$ is said to be nonexpansive if $||f(x)-f(y)|| \leq ||x-y||$ for all x and y in K, and is said to be ultimately nonexpansive if it is continuous and $\limsup_n ||f^n(x)-f^n(y)|| \leq ||x-y||$. If, for each x and y, there is an N such that $||f^n(x) - f^n(y)|| \leq ||x-y||$ for $n > N$, then f is said to be asymptotically nonexpansive.

Note that nonexpansive $\Rightarrow$ asymptotically nonexpansive $\Rightarrow$ ultimately nonexpansive.

There is a considerable amount of literature on nonexpansive mappings, especially with regards to the existence of fixed points. We refer the reader to Kirk [8] for a survey of these results. In general, in "nice" spaces (notably uniformly convex and those with normal structure) every nonexpansive self mapping of a weakly compact convex set has a fixed point.

Ultimately and asymptotically nonexpansive functions have not proven as productive with regards to fixed points. Some of the earlier theorems on nonexpansive mappings have been generalized to asymptotically (cf.[7]) and ultimately (cf.[4]) nonexpansive mappings. These

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