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ON THE PICARD GROUP OF A COMPACT COMPLEX NILMANIFOLD-II

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ABSTRACT. Let G be a complex simply connected nilpotent Lie group and Γ be a lattice subgroup of G. Then the compact complex nilmanifold G/Γ fibres holomorphically over the complex torus $T = G/[G, G]/\pi(\Gamma)$ where $\pi: G \to G/[G, G]$ denotes the quotient map and the fibre is the nilmanifold $[G, G]/\Gamma \cap [G, G]$. Let $\mathrm{Pic}(G/\Gamma)$ denote the Picard group of G/Γ. Then under certain assumptions on T, we are able to obtain a partial generalization of the classical Appell-Humbert Theorem, and in addition, describe $\mathrm{Pic}(G/\Gamma)$ in terms of $\mathrm{Pic}(T)$. Many detailed examples are presented illustrating the nature of G/Γ and its Picard group. See pages 631–638 of the Rocky Mountain J. of Math. Vol. 13, Number 4, Fall 1983 for previous results on this subject.

1. Introduction. Wang [8] showed that compact complex parallelizable manifolds are homogeneous spaces up to analytic equivalence. As interesting examples of such spaces, consider the coset spaces G/Γ where G is a complex simply connected nilpotent Lie group and Γ is a lattice in G. The nilmanifold G/Γ is a natural generalization of the complex torus. Moreover, from the analytic point of view, such spaces provide natural examples of non-Kähler manifolds. In fact, G/Γ is Kähler if and only if it is a complex torus. Further, any such G/Γ has a canonically associated complex torus T given by

$$T = G/[G, G]/\pi(\Gamma) \tag{1.1}$$

where $\pi(\Gamma)$ is a lattice in the vector space $G/[G, G]$ and $\pi: G \to G/[G, G]$ denotes the quotient map. In fact, G/Γ fibres holomorphically over T with fibre the nilmanifold $N_1 = [G, G]/\Gamma_1$, $\Gamma_1 = \Gamma \cap [G, G]$. Let $(G/\Gamma, \pi, T, N_1)$ denote this fibration. See [6] and [7] for details.

This paper deals mainly with the Picard group of G/Γ, denoted $\mathrm{Pic}(G/\Gamma)$. Specifically, we extend some earlier results presented in [2]. As per habit, $\mathrm{Pic}(G/\Gamma)$ is the group of isomorphism classes of holomorphic line bundles on G/Γ. Under a certain condition (see Proposition 2.1), we construct holomorphic maps of T into G/Γ, and we use these same

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