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ON VALUATION RINGS

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ABSTRACT. In this note several definitions and results concerning valuation domains are extended to valuation rings. First we define immediate extension, maximally complete, pseudo-convergent sequence and maximal. It is demonstrated that maximal is equivalent to linearly compact and that maximal implies maximally complete. Note the three are equivalent in the case of valuation domains. We conclude by establishing the existence of a maximal completion of an arbitrary valuation ring.

Notation and Terminology. All rings will be commutative with identity. By a valuation ring we mean a ring whose ideals are linearly ordered by set inclusion. The principal ideals of a valuation ring form a totally ordered monoid where the ordering is reverse inclusion. This monoid will be called the value monoid. (See [**6**].)

We assume throughout that $V(V')$ is a valuation ring with value monoid M (M'), maximal ideal $\mathscr{M}$ ($\mathscr{M}'$) and residue field $k(k')$. We use $\subset$ for proper containment. $U(R)$ will denote the units of a ring R.

1. Maximal, Linearly Compact, Maximally Complete. V' is called an extension of V provided $V \subseteq V'$ and $\mathscr{M}' \cap V = \mathscr{M}$. V' is an immediate extension of V if V' is an extension of V and $k = k'$, $M = M'$. (Equality means the natural embeddings are bijections.) V is called maximally complete if V has no proper immediate extensions. V' is called a maximal completion of V if V' is an immediate extension of V and V' is maximally complete. These definitions stated for valuation domains can be found in [**1**, p. 91], [**5**, p. 36] or [**2**, p. 305].

A set of elements $\{r_\rho\}_{\rho\in A}$ from V is called a pseudo-convergent sequence provided A is a well-ordered set without a last element and $(r_\tau - r_\sigma)V \subset (r_\sigma - r_\rho)V$ for $\rho < \sigma < \tau$. If $\{r_\rho\}$ is such a sequence, then $(r_\sigma - r_\rho)V = (r_{\rho+1} - r_\rho)V$ for all $\rho < \sigma$. Also, either $r_\sigma V \subset r_\rho V$ for all $\rho < \sigma$ or there exists $\lambda \in A$ such that $r_\rho V = r_\sigma V$ for all $\rho, \sigma > \lambda$.

Given a pseudo-convergent sequence $\{r_\rho\}_{\rho\in A}$, for each $\rho \in A$ set $J_\rho = (r_{\rho+1} - r_\rho)V$. An element $r \in V$ is called a pseudo-limit of $\{r_\rho\}$ provided

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