

SMOOTHNESS TO THE BOUNDARY OF CONFORMAL MAPS

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1. Introduction. Usually a genuinely good theorem finds its way from the technical formulations and generally cumbersome proofs of the research literature into books. In the hands of generations of bookwriters, a once forbidding theorem becomes an old friend whose proof is, if not simple, then at least polished, elegant, and compelling. Examples of this phenomenon in analysis are the Riesz Representation Theorem, Picard's Little Theorem, and Mergelyan's Theorem.

The subject of the present paper is an anomaly when viewed in the above light. In spite of the fact that it is a central, significant, and easily formulated result which bears directly on any first course in complex analysis, few people seem to be familiar with it. Fewer still know how to prove it, and there seems to be no optimal or canonical proof. Before we state the result in question we recall that the Riemann Mapping Theorem asserts that if $D \subseteq \mathbb{C}$ is simply connected and not equal to all of \mathbb{C} and if $\Delta \subseteq \mathbb{C}$ is the unit disc then there exists a biholomorphic (one-to-one, onto, holomorphic) mapping $f: \Delta \rightarrow D$. Now we have

THEOREM A. *Let $D \subseteq \mathbb{C}$ be a bounded, simply connected domain which is bounded by a C^∞ smooth Jordan curve. If $f: \Delta \rightarrow D$ is any biholomorphic mapping then f and all its derivatives have continuous extensions to the closure of Δ . Furthermore, f^{-1} and all its derivatives have continuous extensions to the closure of D .*

COROLLARY. *If D_1, D_2 are bounded, simply connected domains with C^∞ boundary and $f: D_1 \rightarrow D_2$ is biholomorphic then f and all its derivatives extend continuously to \bar{D}_1 .*

The proof of the corollary is almost immediate. For the Riemann Mapping Theorem guarantees the existence of biholomorphic maps $\varphi_j: \Delta \rightarrow D_j$, $j = 1, 2$. Theorem A guarantees that each φ_j and all its derivatives extend continuously to $\bar{\Delta}$. Likewise, each φ_j^{-1} and all its derivatives extend continuously to \bar{D}_j . Finally

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