ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 1, Winte 1987

THE ARITHMETIC RING AND THE KUMMER RING OF A COMMUTATIVE RING

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The Witt ring of a commutative ring is a functorial construction which: (1) gives a commutative ring for a commutative ring; (2) has nontrivial value at the field **Q**, or at any number field; and (3) has value at **Q**, or a number field, which is equivalent to a basic circle of successful ideas from classical number theory (see [**5**] and its references). The purpose of this note is to package another problem of classical number theory in this way.

We begin with a general construction, then define what we call the "Kummer ring", $K(R)$, and finally define what we call the "arithmetic ring", $A(R)$. For the special case of R a field whose multiplicative group has an element of order n, for all positive integers n, $A(R)$ is naturally isomorphic to $K(R)$, by the Merkurev-Suslin theorem ([**6**]). We use "ring" (respectively "ring homomorphism") to mean "ring with one" (respectively "ring homomorphism taking one to one").

Let m be a nonnegative integer.

Let X, Y, Z be functors from the category of commutative rings to the category of $(\mathbf{Z}/m\,\mathbf{Z})$-modules. For each commutative ring R, suppose we have a $(\mathbf{Z}/m\,\mathbf{Z})$-bilinear map

$$\phi_R: X(R) \times Y(R) \to Z(R)$$

which is functorial in R. By this we mean, if $f: R \to k$ is a homomorphism of commutative rings, then

$$Z(f)\,(\phi_R(x, y)) = \phi_k(X(f)(x),\ Y(f)(y)),$$

for all $x \in X(R)$, $y \in Y(R)$. First let $m = 0$. We define $M(R)$ to be

$$\mathbf{Z} \times X(R) \times Y(R) \times Z(R).$$

We define operations on $M(R)$ by

$$(n_1, x_1, y_1, z_1) + (n_2, x_2, y_2, z_2) = (n_1 + n_2, x_1 + x_2, y_1 + y_2,\\ z_1 + z_2 + \phi_R(x_1, y_2) + \phi_R(x_2, y_1)),$$

Received by the editors on February 20, 1985.