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TANGENT BALL EMBEDDINGS OF SETS IN E^3

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R. H. Bing [3] and M. K. Fort, Jr. [18] asked if a 2-sphere in E^3 would be tamely embedded in E^3 if it were known to have double tangent balls on opposite sides at each of its points. A 2-sphere Σ in E^3 is said to have double tangent balls at a point p of Σ if there exist two round 3-cells B and B' such that $B \cap B' = \{p\} = \Sigma \cap (B \cup B')$. The balls B and B' are on opposite sides of Σ if they lie, except for p, in different components of $E^3 - \Sigma$. Appearing shortly after publication of Bing's 1-ULC characterization of tame surfaces [1] and about the same time as his Side Approximation Theorem [2] the question, whose answer depended upon these famous theorems, led to many related embedding facts. In this note I will summarize the evolution of "tangent ball theory" as it developed from the Bing/Fort Question. The account is historical with no proofs given, but some of the facts related here have not appeared elsewhere.

In both [3] and [18] the double tangent ball question arose in the context of piercing wild spheres with geometrically nice objects. Since Bing [3] and Fort [18] produced examples of wild 2-spheres that could be pierced with segments, it was natural to ask whether more general geometrical piercing sets such as balls and cones would be sufficient to insure the tameness of a surface. It is doubtful that either researcher was trying to produce a geometric analogue to smoothness as mildly suggested in [15] and [16]. Still it seems appropriate to give some examples to show there is no direct relation between the existence of double tangent balls and the surface having continuously differentiable defining functions. Let $f(x)$ be a function much like $x^2 \sin 1/x$ except adjusted slightly, if necessary, so that its graph in E^2 has double tangent disks at every point. The surface Σ_1, obtained by rotating this graph about the z-axis, is not continuously differentiable at the origin, yet Σ_1 has double tangent balls at each of its points. On the other hand let Σ_2 be the surface obtained by rotating the graph of $g(x) = x^{3/2}(0 \leqq x \leqq 1)$ about the z-axis. This surface has continuously differentiable coordinate functions but, with some effort, one can show there do not exist double tangent balls to Σ_2 at the origin. However a surface will have double tangent balls if it has sufficiently smooth coordinate functions [27].

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