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A COMPUTER METHOD FOR APPROXIMATING THE ZEROS OF CERTAIN ENTIRE FUNCTIONS

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1. Introduction. The computer revolution has greatly enhanced techni ques used in numerical calculations for a wide range of functions. This paper investigates some properties of a certain class of entire functions known as the Lindelöf functions. These functions are of the form:

$$f(z) = \prod_{N=1}^{\infty}\left(1 - \frac{z}{N^A}\right), \ A > 1, \ z \text{ complex}. \tag{1}$$

Since $f(z)$ is an infinite product, it is a generaliziation of a polynomial, and hence has a special appeal. In addition, when $A = 2$, we have the well-known equality

$$\frac{\sin \pi \sqrt{z}}{\pi \sqrt{z}} = \prod_{N=1}^{\infty}\left(1 - \frac{z}{N^A}\right).$$

Thus these functions are also intimately related to the trigonometric functions.

In 1972 King and Shah (cf. [3]) exhibited selected properties of the Lindelöf Functions. As a partial proof of one theorem, it was necessary to approximate the bounds for the zeros of the derivatives of these functions. The techniques were slow and cumbersome due to the lack of computer facilities available to the authors at that time. We present here a method for locating the bounds, which is not only accurate to several decimal places, but greatly improves the previous values obtained for these bounds. A proof of a required theorem is given, along with sample computer calculations illustrating the method, and a discussion of the results.

2. Mathematical analysis. We shall examine the zeros of $f'(z)$, the derivative of the Lindelöf Function, $f(z) = \prod_{N=1}^{\infty} (1 - z/N^A)$, $A > 1$. These Lindelöf Functions are a special subset of Functions of Bounded Index [2]. Both f and all its derivatives are of Bounded Index [3]. From a theorem of Laguerre (cf. [1]) the zeros of $f'(z)$ are all real and are separated by the

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