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SUMMING SUBSEQUENCES OF RANDOM VARIABLES

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ABSTRACT. Given an increasing sequence N of positive integers and $k \geqq 1$, call any one to one correspondence $\tau: N \to \mathbf{N}^k$ an ordering (or numbering) of N onto $\mathbf{N}^k$. Let (X_n) be a sequence of random variables satisfying $\sup_n \mathrm{E}\,|X_n|\,(\log^+ |X_n|)^{k-1} < \infty$. Then there exists a subsequence $N_0 = (i_n)$ such that, for any further subsequence $N_1 = (i_{j_n})$ and any ordering τ satisfying $|\tau(i_{j_n})| \leqq j_n$ for all $n \geqq 1$, we have $(X_{\tau^{-1}(s)})$ converges Cesàro a.s. for $s \in \mathbf{N}^k$.

1. Introduction and notation. The theorem of Komlós [2] is a generalized strong law of large numbers. If (X_n) is an L_1-bounded sequence of random variables, then there exists a subsequence such that every further subsequence converges Cesàro a.s., to the same limit. In this paper, the following Komlós-type property is considered. Given a sequence (X_n) satisfying a certain moment condition, there exists a subsequence (X_n^0) such that any ordering, to a degree, of any subsequence of (X_n^0) into $\mathbf{N}^k$ converges Cesàro a.s. The limit is independent of the particular subsequence of (X_n^0), and of the ordering. As a corollary (taking $k = 1$), to a large degree, permutations of the Komlós subsequences converge Cesàro a.s.

This latter result cannot be obtained from Komlós's proof, which uses martingale difference sequences. The method used here is patterned after Etemadi's [1] proof of the strong law of large numbers for pairwise independent, identically distributed random variables. Despite the fact that we begin with a sequence (X_n) rather than an array, the moment condition must be stronger than L_1-bounded to obtain the result; we suppose $\sup_n \mathrm{E}|X_n|(\log^+|X_n|)^{k-1} < \infty$. This condition is not always necessary, but Smythe [4] has shown that if $\mathrm{E}|X_n|(\log^+|X_n|)^{k-1} = \infty$, then the strong law of large numbers fails to hold for a k-dimensional array of i.i.d. random variables. Consequently, a multiparameter Komlós-type theorem cannot hold in general if (X_n) is only L_1-bounded.

In the following, let (X_n) be a sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$. For $k \geqq 1$, we consider $\mathbf{N}^k$ with the coordinatewise partial ordering $\leqq$. For $s = (s_1, \ldots, s_k) \in \mathbf{N}^k$, denote $|s| = s_1 \cdot \ldots \cdot s_k$. If $j \geqq 1$, let $d_j = \operatorname{card}\{s \in \mathbf{N}^k : |s| = j\}$, the number of ways of writing

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