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IMAGES AND QUOTIENTS OF SO(3, R): REMARKS ON A THEOREM OF VAN DER WAERDEN

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1. Introduction. We present an elementary proof of the following special case of a general theorem of B. L. van der Waerden: Every homomorphism from the rotation group $SO(3, \mathbf{R})$ to a compact topological group is continuous. An equivalent property of $SO(3, \mathbf{R})$, whose analogue is false for every infinite, compact, Abelian group, is this: No totally bounded topological group topology for $SO(3, \mathbf{R})$ is finer than the usual topology. The proof follows from a lemma which has an additional consequence: algebraically, the group $SO(3, \mathbf{R})$ is simple.

2. Background. In the context of (Hausdorff) topological groups, a powerful algebraic property—namely, the property that the group in question be Abelian—can have powerful topological consequences. Here are two examples of what we have in mind. (1) Every infinite Abelian group admits a totally bounded topological group topology **[14]**, **[6]**, but for non-Abelian groups the corresponding statement fails **[8]** (p. 296 ff.), **[17]** (p. 157), **[20]**, **[30]**, **[12]** (pp. 348–351). (2) Every infinite Abelian group admits a non-discrete metrizable topological group topology **[16]**, **[6]**, **[23]**, but there are non-Abelian groups which become topological groups only under the discrete topology **[24]**, **[1]** (§ 13.4).

The present paper originates with a question of much the same flavor: Can a compact topology on an infinite group be maximal among totally bounded topological group topologies? A simple argument (which we record below), based on classical cardinality constraints, shows that for Abelian groups the answer is "No". There exist, however, non-Abelian topological groups which show that the answer is "Yes". This is essentially a result, over a half-century old, due to B. L. van der Waerden **[28]**; as our Introduction suggests, our approach is via consideration of the existence of discontinuous homomorphisms into compact groups.

Our goals in this paper are to identify and describe the modern setting and vocabulary appropriate to van der Waerden's theorem, to provide an elementary proof from first principles of the result as it concerns specifically the compact group $SO(3, \mathbf{R})$, and to record some consequences of the theorem which are not mentioned explicitly by van der Waerden.

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