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A REMARK ON THE ENERGY OF HARMONIC MAPS BETWEEN SPHERES

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1. Introduction. A harmonic map between two Riemannian manifolds is a critical point of the energy integral. The conformal invariance of this integral in two dimensions makes this variational problem especially tractable. A fact that is special to two dimensions is that a harmonic map $\phi: S^2 \to S^2$ is energy minimizing among all C^2 maps homotopic to ϕ. Furthermore, it is well known that the energy of a harmonic map $\phi: S^2 \to S^2$ is given by

$$E(\phi) = \frac{1}{2}\int_{S^2} |d\phi|^2 dv = |\deg \phi| \mathrm{vol}(S^2). \tag{1}$$

In contrast to the two dimensional situation, Eells and Sampson [2] showed that any differentiable map $\phi: S^n \to S^n$ ($n \geqq 3$) of nonzero degree does not minimize energy within its homotopy class. It is then natural to ask if there exist stable, harmonic maps $\phi: S^n \to S^n$ when $n \geqq 3$. This question was answered in the negative by Y.L. Xin [5] who proved the following more general theorem.

THEOREM. (XIN). *If $n \geqq 3$, there exists no nonconstant, stable harmonic map from S^n to any Riemannian manifold.*

Xin proved this result by computing the second variation of the energy along the conformal vector fields of S^n. A conformal vector field on S^n is of the form $\nu = \mathrm{grad}(\lambda|_{S^n})$, where λ is a linear functional on $\mathbf{R}^{n+1}$. Let $\phi_t: S^n \to M$ be a one parameter variation of a harmonic map $\phi = \phi_0$ such that

$$\left.\frac{d\phi_t}{dt}\right|_{t=0} = \phi_*(\nu), \tag{2}$$

where ν is a conformal vector field on S^n. Xin proves

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