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BOUNDEDNESS FOR BLOCH FUNCTIONS

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ABSTRACT. Two theorems concerning the boundedness for certain functions in the Bergman space for the unit disc are proven. Theorem 1. If f is in the Bergman space so that $|f(z)| \leqq m$ for all z in the crescent region bounded by $|z| < 1$ and $|z-x| < 1-x$, $0 < x \leqq 1/2$, then $|f(z)| \leqq m$ for all z in the unit disc. Theorem 2. If f is a Bloch function so that $\limsup_{z \to a} |f(z)| \leqq m$ for all but a finite number of a's in the boundary of the unit disc, then $f(z)$ is bounded on the unit disc.

Introduction. The Bergman p-space for the open unit disc Δ is the closure of the analytic functions in $L^p(\Delta, dA)$ where dA is area measure. In this paper the relationships between integrability and boundedness on Δ will be investigated. Let $A_p(\Delta)$ denote the Bergman p-space, $p \geqq 1$.

It is clear (maximum modulus theorem) that if $f \in A_1(\Delta)$ and f is bounded on the annular region bounded by $|z| = 1$ and $|z| = r$, $r < 1$, then f is bounded on Δ. However, for $f \in A_1(\Delta)$ and f bounded on the open crescent region bounded by $|z| = 1$ and $|z - x| = 1 - x$ for $0 < x \leqq 1/2$, it is not clear that $f(z)$ is bounded on Δ. This will be shown to be true as a corollary of a stronger result for crescent regions.

This result represents the interplay between the maximum modulus theorem and integrability. It is conjectured by the author that if $f \in A_1(\Delta)$ and $\limsup_{z \to a} |f(z)| < M$ for all but a finite number of points $a \in \partial\Delta$, then f is bounded. This conjecture will be shown to be true for the space of Bloch functions for Δ.

Notations & Definitions. Throughout this paper Δ will be used for the open unit disc and G will be the crescent region bounded by $|z| = 1$ and $|z - x| = 1 - x$ where $0 \leqq x \leqq 1/2$. Let $U = \Delta/\bar{G}$. Then ∂U is parametrized by $\Gamma(\theta) = x + (1 - x)e^{i\theta}$ where $0 \leqq \theta < 2\pi$. The closure of the analytic polynomials in the Bergman p-space for G will be denoted by $H_p(G)$. The standard Hardy p-space for the unit circle will be given by $H_p(\partial\Delta)$.

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