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ON THE SPACES CLASSIFYING COMPLEX VECTOR BUNDLES WITH GIVEN REAL DIMENSION

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ABSTRACT We compute the integral cohomology ring of the space classifying complex vector bundles of real geometric dimension at most n and generalise this to any complex oriented theory; also we rederive an integrality condition of Astey and Gitler for certain K-theory characteristic classes of such bundles and relate these to a "universal unit" of Ray, Switzer, Taylor.

Introduction. We will compute the integral cohomology ring of the space BU_n which classifies complex vector bundles of real geometric dimension at most n. The form of the result depends on the parity of n but in each case there is neither torsion nor non-zero odd degree cohomology. In particular our results give polynomial generators well related to the Chern classes of the canonical complex bundle over BU_n. We generalise this to an arbitrary "complex oriented" cohomology theory $E^*(\ )$ (e.g., $MU^*(\ )$, $KU^*(\ )$); the method we use for this involves calculating the E-homology of the Bott space SO/U and the construction of a dual basis in E-cohomology. Finally we consider a specific element in $KU^\circ(BU_n)$ which has been used by L. Astey and S. Gitler to derive non-sectioning results for bundles; we also explain the relation between this and a "universal unit" of [**12**].

The results of §1 and §2 are contained in the author's 1980 Ph.D. thesis and an earlier preprint (October 1980). §4 contains results found after conversations with S. Gitler. There is some overlap with the results of [**6**], in particular the idea of the proof of *Theorem* (*2.2*) is the same although we give our version to highlight certain details we require for later use.

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1. For any "stable subgroup" G of the infinite special orthogonal group SO we have G vector bundles and virtual bundles defined using the existence of inclusions (assumed as part of the data)

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