ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 4, Fall 1986

EXTREMA AND NOWHERE DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We give simple, unified constructions of continuous, nowhere differentiable functions that (1) have no proper local maxima or proper local minima, (2) have no proper local maxima but have proper local minima at every point of a dense set, and (3) have proper local maxima at every point of a dense set, and proper local minima at every point of another dense set.

1. Introduction. A function $f: \mathbf{R} \to \mathbf{R}$ is said to have a proper (or strict) local maximum at p in $\mathbf{R}$ provided there exists $\varepsilon > 0$ such that if $0 < |x - p| < \varepsilon$, then $f(x) < f(p)$. A function $f: \mathbf{R} \to \mathbf{R}$ is said to have a local maximum at $p \in \mathbf{R}$ provided there exists $\varepsilon > 0$ such that if $|x - p| < \varepsilon$, then $f(x) \leqq f(p)$. The terms proper local minimum and local minimum are defined in the obvious way. The main purpose of this paper is to give simple, unified constructions of the following examples.

EXAMPLES 1.1. There exist continuous, nowhere differentiable, real valued functions f, g, and h of a real variable such that

A. The function f has no proper local maxima and no proper local minima, and, furthermore $f^{-1}(y)$ is a perfect subset of $\mathbf{R}$ for every $y \in \mathbf{R}$.

B. The function g has no proper local maxima, but has proper local minima at every point of a dense subset of $\mathbf{R}$.

C. The function h has proper local maxima at every point of a dense subset of $\mathbf{R}$ and has proper local minima at every point of another dense subset of $\mathbf{R}$.

We also mention the following result which concerns *all* local extrema of continuous, nowhere monotone functions. In particular, it applies to the function f of Example A.

THEOREM 1.2. *If $f: \mathbf{R} \to \mathbf{R}$ is a continous function which is not monotone over any interval, then the set of points where f has a local maximum and the set of points where f has a local minimum are both dense in* $\mathbf{R}$. *Further, both of these sets are sets of first category.*

Subject classification: 26A15, 26A27, 26A30, 54C05.
Key words and phrases: continuity, nowhere differentiable, proper (strict) local maxima and minima, local maxima and minima, first category.
Received by the editors on April 9, 1984 and in revised form on Jaunary 28, 1985.