ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 3, Summer 1986

BEZIER-CURVES WITH CURVATURE AND TORSION CONTINUITY

HANS HAGEN

ABSTRACT. One of the main problems in computer-aided design is how to input shape information to the computer. In the analytic description and approximation of arbitrary shaped curves the Bezier-curves are of great importance (see [5]). A Bezier-curve is a segmented curve. The segments $x_\ell(u) := \sum_m^{i=0} b_{m\ell+i} \cdot B_i^m(u - u_\ell / u_{\ell+1} - u_\ell)$ of a Bezier-curve of degree m over the parameter interval $u_\ell \leqq u \leqq u_{\ell+1}$ use the Bernstein-polynomials as blending functions. The coefficients $b_{m\ell+i}$ are called Bezier points. They form the so called Bezier polygon, which implies the Bezier-curve.

A.R. Forrest analyzed the Bezier techniques in [4] and extended these techniques to generalized blending functions.

W. J. Gordon and R. F. Riesenfeld provided in [5] an alternative development in which the Bezier methods emerge as an application of the Bernstein polynomial approximation operator to vector-valued functions.

As connecting conditions between the curve-segments are always chosen the so called C^2 – or C^3 – continuity. (A segmented curve is said to have $C^{(k)}$-continuity if an only if $X^{(k)}(t_i^+) = X^{(k)}(t_i^-)$ at the connecting points t_i; $i = 1, \ldots, n$, where $X^{(k)} := (\partial/\partial t^k)X$; $k \in N$.)

In this paper we create, after a brief survey of the fundamentals of differential geometry, a tangent, a curvature, and a torsion continuity, using the geometric invariants of a curve.

Considering $C^2-(C^3-)$ continuity, we have only one choice for $b_{m(\ell+1)+2}(b_{m(\ell+1)+3})$, $0 = \ell \leqq k$. In the third part of this paper we show that curvature continuity offers a "straight line of alternatives" and torsion continuity offers a "plane of alternatives."

We give also constructions for the Bezier polygons of Bezier curves with curvature – and torsion – continuity, which are convenient for a graphic terminal.

1. Fundamentals of differential geometry.

DEFINITION 1.1. (a) A parametrized C^r-curve is a C^r-differentiable map $X: I \to E^n$ of an open interval I of the real line R into the euclidean space E^n.

(b) A parametrized C^r-curve $X: I \to E^n$ is said to be regular if $\dot{X}(t) \neq 0$, for all $t \in I$, where $\dot{X} = \partial/\partial t X$.

Received by the editors on November 13, 1983 and in revised form on January 11, 1985.