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ON THE NUMBER OF MINIMAL PRIME IDEALS IN THE COMPLETION OF A LOCAL DOMAIN

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Let R be a local Noetherian domain. It is well-known that the number of minimal prime ideals in the completion of R is greater than or equal to the number of maximal ideals in the integral closure of R. An (unproved) exercise in [2] states that the reverse inequality holds if R is one-dimensional. The purpose of this note is to show how this latter fact can be generalized to local domains of dimension greater than one. Specifically, let $x_1, \ldots, x_d$ be a system of parameters for R and set

$$T = R\left[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right]_{MR[\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}]}$$

(M is the maximal ideal of R). We will show that if R is quasi-unmixed, then the number of maximal ideals in the integral closure of T is greater than or equal to the number of minimal prime ideals in the completion of R. As a corollary we deduce a criterion for local domains to be analytically irreducible and we close with a bound for the number of minimal prime ideals in the completion of R in the non-quasi-unmixed case.

NOTATION. Throughout, (R, M) will denote a local Noetherian ring with maximal ideal M. We will use "—" to denote integral closure—both for rings and ideals. Recall that for an ideal $I \subseteq R$, $\bar{I}$, the integral closure of I, is the set of elements $x \in R$ satisfying an equation of the form

$$x^n + i_1 x^{n-1} + \cdots + i_n = 0, \quad i_k \in I^k, \ 1 \leqq k \leqq n.$$

It is well-known that $\bar{I}$ is an ideal of R contained in the radical of I. We will use "*" to denote the completion of a local ring. Recall that a local ring R is quasi-unmixed in case $\dim R^*/p^* = \dim R$, for all minimal primes $p^* \subseteq R^*$. Any other standard facts or terminology from local ring theory appear here as they do in [2].

REMARK. Lemmas 1 and 2 below are more or less well-known, but we have included their easy proofs for the sake of exposition.

LEMMA 1. (c.f. [6, p. 354]): *Let R be a Noetherian domain and $I \subseteq R$*