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AFFINE GEOMETRY AND THE FORM OF THE EQUATION OF A HYPERSURFACE

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Introduction. In classical geometry, a subset M of $\mathbf{R}^{n+1}$ is said to be a hypersurface if it is the zero-set of some (appropriately restricted) function. This function is not uniquely determined by M; for example, if M is the zero-set of F: $\mathbf{R}^{n+1} \to \mathbf{R}$, then it is also the zero-set of $G = h \cdot F$ for any nowhere-vanishing function h: $\mathbf{R}^{n+1} \to \mathbf{R}$. In other words, the "equation of M" can assume many different forms.

QUESTION. Is there a "canonical form" for the equation of M?

We do not answer this question here, but we do single out a class of "preferred forms" for the equation of any nondegenerate hypersurface $\mathbf{R}^{n+1}$. (See §1.)

The preferred forms for the equation of M give rise to certain geometric objects. Prescribing a volume element on $\mathbf{R}^{n+1}$ normalizes these objects, which then turn out to be well-known quantities from affine geometry: the Berwald-Blaschke (affine) metric, the Fubini-Pick form, and the affine normal. Our approach to these quantities is coordinate-free and seems simpler than the standard treatments which focus on the special linear group (see, for example, Blaschke [**2**], Guggenheimer [**5**], or Spivak [**6**]). (In particular, our approach does not require verifying the invariance of these quantities under change of parameters, since no parametric representation is used in the definitions.) Indeed, this paper could be used as a quick introduction to the basic notions of affine hypersurface-geometry.

Much of our formalism makes sense in spaces of infinite dimension, and our main result (Theorem C, in §2) characterizes the nondegenerate quadratic hypersurfaces in a Banach space. (The finite-dimensional version is Berwald's theorem [**1**] in affine geometry.)

Only minor changes are needed to make our discussion applicable to arbitrary level sets (and not just to zero-sets). In §3 we briefly consider those functions F such that on each level set of F, the equation $F = c$ describing that set is one of the "preferred forms" for the equation of that

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