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INDECOMPOSABLE MODULES CONSTRUCTED FROM LIOUVILLE NUMBERS.

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ABSTRACT. The submodules of the polynomial Kronecker module are investigated. A pair of vector spaces (V, W) over an algebraically closed field K is called a Kronecker module if there is a K - bilinear map form $K^2 \times V$ to W. Every module over $K[\xi]$ - the polynomial ring in one variable over K - may be viewed as a Kronecker module. The polynomial Kronecker module **P**, is $K[\xi]$ so viewed. Every infinite-dimensional submodule of **P** of finite rank has a unique infinite-dimensional indecomposable direct summand. So attention is focussed on indecomposable submodules. In that direction the main result is: For each positive integer $n > 1$, there is a family $\{V_s: s \in S\}$, Card $S = 2^{\aleph_0}$, of indecomposable submodules of **P** of rank n with the following properties:

(a) $\mathrm{Hom}(V_{s_1}, V_{s_2}) = 0$ if $s_1 \neq s_2$;

(b) $\mathrm{End}(V_s) = K$ for every s in S;

(c) $\dim \mathrm{Ext}(V_{s_1}, V_{s_2}) \geqq 2^{\aleph_0}$ for any s_1, s_2 in S.

This result is proved by constructing extensions of finite-dimensional modules by **P** using Liouville numbers. Each extension, **V**, is shown to share with **P** a common submodule which reflects properties of **V**. A consequence of this is that, for each positive integer $n > 1$, **P** contains a nonterminating descending chain of nonisomorphic indecomposable submodules of rank n.

1. Completely decomposable submodules of P. Throughout the paper K is a fixed algebraically closed field and (a, b) is a fixed basis of the two-dimensional K-vector space K^2. Since the map from $K^2 \times V$ to W is bilinear it is enough to specify it on (a, b) and a basis of V. In $P = (K[\xi], K[\xi])$ the bilinear map is given by $af = f$, $bf = \xi f$ for all polynomials f.

Each $e \in K^2$ gives rise to a linear transformation $T_e: V \to W$ defined by $T_e(v) = ev$, the image of (e, v) under the bilinear map from $K^2 \times V$ to W. If T_e is one-to-one for every nonzero e in K^2, **V** is said to be torsion-free. So P is torsion-free. Observe that P is an ascending union, $\bigcup_{k=1}^{\infty} \mathbf{V}_k$, of finite-dimensional submodules where $\mathbf{V}_1 = (0, [1])$; and, for $k \geqq 2$,

$$(1) \qquad V_k = [1, \xi, \ldots, \xi^{k-2}],\ W_k = [1, \ldots, \xi^{k-2}, \xi^{k-1}].$$

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