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A SIMPLE PROBLEM FOR THE SCALAR WAVE EQUATION ADMITTING SURFACE-WAVE AND AH-WAVE SOLUTIONS

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It has recently been found that the surface- and AH-wave solutions of a classical problem for Maxwell's equations [1] are generated by solutions of a simple problem for the scalar wave equation in R^3, namely,

$$
\begin{aligned}
&(\partial_t^2 - c_0^2\Delta)\phi(x, t) = 0,\ x_3 > 0,\ t > 0\\
&(\partial_t^2 + \kappa\partial_t - c^2\Delta)\phi(x, t) = 0,\ x_3 < 0,\ t > 0,\\
&\phi(x, 0^+) = f(x),\ \partial_t\phi(x, 0^+) = F(x),\\
&c^2\partial_3\phi(x', 0^-, t) = c_0^2\partial_3\phi(x', 0^+, t),\\
&\partial_t\phi(x', 0^-, t) + \kappa\phi(x', 0^-, t) = \partial_t\phi(x', 0^+, t),
\end{aligned} \tag{1}
$$

where $c_0 > c > 0$ and $\kappa > 0$ are constants and $x' = (x_1, x_2)$. In the present note we show that problem (1) is uniquely solvable for a certain class of intial data (f, F) and present the explicit form of the surface- and AH-wave solutions to (1). We further show how to construct the corresponding solutions to the classical problem for Maxwell's equation (2) from the solutions to (1). Surface-wave solutions of (1) are superpositions of modes with frequencies having nonzero real and imaginary parts which decay exponentially in space away from the interface $\{x_3 = 0\}$. AH-wave solutions are super-positions of modes having this same spatial decay, but their frequencies have no real part, so they simply decay in time without propagating—rather peculiar wave-like behavior.

Denoting by $\chi_\pm = \chi_\pm(x_3)$ the characteristic functions of the half spaces $R^3_\pm = \{x \in R^3\colon \pm\ x_3 > 0\}$ and defining the 6×6 diagonal matrices $E_\pm = \mathrm{diag}[\varepsilon_\pm I_3, \mu_\pm I_3]$, $B = \mathrm{diag}(\sigma I_3, 0_{3\times 3})$, where I_3 is the 3×3 identity matrix, the Cauchy problem for Maxwell's equations in two semi-infinite media (the lower of which is conducting) separated by the plane boundary $\{x_3 = 0\}$ can be written

$$
\begin{aligned}
\partial_t f(x, t) = \{&\chi_+(x_3)E_+^{-1} A(\partial)\\
&+ \chi_-(x_3)E_-^{-1}[A(\partial) + B]\}f(x. t),\ x_3 \neq 0,\ t > 0,\\
f(x, 0^+) = &f_0(x) \in L_2(R^3, C^6),
\end{aligned} \tag{2}
$$

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