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ANOTHER PROOF OF EISENSTEIN'S LAW OF CUBIC RECIPROCITY AND ITS SUPPLEMENT

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ABSTRACT. A new proof is given of the law of cubic reciprocity and its supplement.

1. Introduction. The domain of Eisenstein integers $x + y\omega$, where x, y are rational integers and $\omega = (-1 + \sqrt{-3})/2$, is denoted by $Z[\omega]$. The domain $Z[\omega]$ is a unique factorization domain and its primes consist of rational primes congruent to 2 (mod 3) and their associates, complex primes of the form $a + b\omega$ with norms $N(a + b\omega) = a^2 - ab + b^2$ equal to rational primes congruent to 1 (mod 3), and $1 - \omega$ and its associates. Each prime, which is not an associate of $1 - \omega$, has exactly one of its six associates which is primary, that is, congruent to 2 (mod 3).

If λ is a prime in $Z[\omega]$, which is not an associate of $1 - \omega$, then the norm of λ is congruent to 1 (mod 3) and the cubic residue character χ_λ is defined for $\alpha \in Z[\omega]$ by

$$\chi_\lambda(\alpha) = \begin{cases} 0, \text{ if } \alpha \equiv 0 \pmod{\lambda}, \\ \omega^r, \text{ if } \alpha \not\equiv 0 \pmod{\lambda} \text{ and } \alpha^{(N(\lambda)-1)/3} \equiv \omega^r \pmod{\lambda}, r=0, 1, 2. \end{cases}$$

In 1844 Eisenstein [3] proved the law of cubic reciprocity.

If λ_1 and λ_2 are primary primes of $Z[\omega]$ with $N(\lambda_1) \neq N(\lambda_2)$ then

$$\chi_{\lambda_1}(\lambda_2) = \chi_{\lambda_2}(\lambda_1). \tag{1.1}$$

In a later paper [4] he proved the supplement to the law of cubic reciprocity which treats the exceptional prime $1 - \omega$.

If λ is a primary prime of $Z[\omega]$ then

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