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MORE q-BETA INTEGRALS

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ABSTRACT. Two new q-extensions of Barnes' beta integral are found. A third one found by Watson is reproved, and in the course of doing this another q-beta integral is discovered.

1. Introduction. Barnes [7] evaluated an integral of the product of four gamma functions which usually goes under the name of Barnes' lemma. This integral is an extension of Euler's classical beta function, so we will call it Barnes' beta integral. When $\mathrm{Re}(a, b, c, d) > 0$ this integral is

$$\begin{aligned}(1.1)\qquad &\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)dt \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.\end{aligned}$$

To see that this extends Euler's beta integral replace b by $b - iw$, d by $d + iw$ and set $t = wx$. Then Stirling's formula can be used to obtain

$$(1.2)\qquad \int_{-\infty}^{\infty} x_+^{a+c-1}(1-x)_+^{b+d-1}\,dx = \frac{\Gamma(a+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

where $x_+ = x$ if $x \geqq 0$, $x_+ = 0$ if $x < 0$.

Watson [12] found a q-extension of (1.1). This will be given in §3. There are others, and the easiest way to find them was given by Titchmarsh [11, pp.193–194]. Here is his proof of (1.1). Take Euler's beta integral on $[0, \infty)$

$$(1.3)\qquad \int_0^{\infty} \frac{t^{\alpha-1}\,dt}{(1+t)^{\alpha+\beta}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and consider it as a Mellin transform. Then use the L^2 theory of Mellin transforms. One of these results is the following. If

$$F_j(x) = \int_0^{\infty} t^{x-1} f_j(t)dt$$

then

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