ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 2, Spring 1986

COMPACTNESS IN SPACES OF GROUP-VALUED CONTENTS, THE VITALI-HAHN-SAKS THEOREM AND NIKODYM'S BOUNDEDNESS THEOREM

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0. Introduction. The starting point of this paper is the following theorem of W.G. Graves and W. Ruess [**11**, Theorem 7].

THEOREM. *For a locally convex space E and a subset K of the space of E-valued measures on a σ-algebra Σ, K is relatively compact in the topology of pointwise convergence (on each $A \in \Sigma$) if and only if $K(A)$ is relatively compact for each $A \in \Sigma$ and K is uniformly σ-additive.*

Graves and Ruess proved this theorem in the setting of Graves theory [**10**] of s-bounded measures with values in a locally convex space, the main idea of which is a topological linearization of the study of such measures, using as central device the "universal measure space" and its topology.

In this paper the theorem mentioned above is proved completely elementarily and generalized for group-valued measures. The essential part of this theorem, namely that the compactness of K implies the uniform s-boundedness, may be considered as a generalization of the Vitali-Hahn-Saks theorem (in the σ-additive case), for which there are elementary, transparent proofs (see, e.g., [**16, 17**]). This part is here proved by a refinement of the methods in the proofs for the Vitali-Hahn-Saks theorem. The proof is carried through in such a way that it yields, without extra work, the Vitali-Hahn-Saks theorem for s-bounded (finitely additive) contents, Nikodym's boundedeness theorem (for contents with values in a quasi-normed group), Rosenthal's lemma, and a criterion for uniform s-boundedness of A.B. d'Andrea de Lucia and P. de Lucia.

The paper is structured as follows. In §2, certain $[0, \infty]$-valued functions on a Boolean ring R are studied. As the main result of this section we get, in Theorem 2.4, a criterion for s-boundedness, from which the compactness criterion mentioned above and the Vitali-Hahn-Saks theorem can be easily deduced. It is of interest that no further assumption for R (like σ-completeness) is needed in Theorem 2.4. In §3.1 we obtain, as a

Received by the editors on February 6, 1984.