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REFLECTION GROUPS AND MULTIPLICATIVE INVARIANTS

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Introduction. Given a lattice M (i.e., a finitely generated torsion free abelian group), one can form the group algebra $\mathbf{C}[M]$. The operation for M, usually thought of as addition, must then be regarded as multiplication. An automorphism of M extends to an algebra automorphism of $\mathbf{C}[M]$ in a unique way. We refer to $GL(M)$ as inducing a "multiplicative action" on $\mathbf{C}[M]$.

The semi-expository paper **[2]** is devoted to such actions. One of the theorems proved there was a multiplicative analogue of the Shephard-Todd-Chevalley Theorem.

THEOREM. *Assume M is a lattice and G is a finite subgroup of $GL(M)$. Then the fixed ring $\mathbf{C}[M]^G$ is a polynomial ring over $\mathbf{C}$ if and only if G is a reflection group and, for some choice of root system, M is isomorphic as a module to a weight lattice over its Weyl group G.*

Subsequently, I was led to a paper of Steinberg **[6]** in which a related theorem appears. Indeed, it is fair to say that the theorem above is implicit in Steinberg's work. Apparently, it has been valuable, for general ring theorists, to bring the invariant theoretic statement into relief. My arguments are naive in the sense that they use no algebraic geometry and employ only the rudiments of root systems.

This note is an elaboration of the second half of **[2]**. The theorem stated above says that, even for reflection groups, it is rare that the fixed ring of the group algebra is a polynomial ring. This distinction among G-module structures for M disappears once we pass to the rational function field of fractions $\mathbf{C}(M)$.

THEOREM 10. *Assume M is a lattice and $G \subset GL(M)$ is a finite reflection group. Then $\mathbf{C}(M)^G$ is always a rational function field.*

The same techniques prove a generalization of the invariant theorem.

COROLLARY 13. *Assume M is a lattice and $G \subset GL(M)$ is a finite re-*

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