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UNIVALENT FUNCTIONS HAVING UNIVALENT DERIVATIVES

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ABSTRACT. For functions of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqq 0$, that are analytic and univalent in the unit disk, we investigate subclasses of functions having some or all of their derivatives univalent. Sufficient conditions are given for the functions to be in the various classes and a sharp upper bound for the second coefficient of functions whose derivatives are all univalent is found. Surprisingly, there is no function in the class whose second coefficient attains this sharp upper lound.

1. Introduction. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in the family S if it is analytic and univalent in the unit disk $\Delta = \{|z| < 1\}$. If f may further be expressed as

$$(1) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geqq 0.$$

then f is said to be in the family T. In **[6]** it is shown that functions of the form (1) are in T if and only if $\sum_{n=2}^{\infty} n\, a_n \leqq 1$. This enabled us to show that the extreme points of T were z and $z - z^n/n$ $(n = 2, 3, \ldots)$.

Denote, by T_1, the subfamily of T consisting of functions f for which f' is also univalent in Δ. Since the second coefficient of a function in T_1 cannot vanish, the only extreme point of T that is also a member of T_1 is $z - z^2/2$. Although T_1, unlike T, cannot easily be characterized by its coefficients, we do find separate sufficient and necessary conditions that lead to various coefficient bounds. We also investigate subfamilies of T for which higher order derivatives are univalent and obtain a sharp upper bound for the second coefficient when all derivatives are univalent.

2. The family T .

THEOREM 1. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1$, *then* $a_2 \leqq 1/2$ *and*

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