ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 1, Winter 1986

EXAMPLES OF RP-MEASURES

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ABSTRACT. We call a measure on the torus T^2 an RP-measure if its Poisson integral is the real part of a holomorphic function. Let RP_1 denote the set of RP-measures which are non-negative and have total mass one. We construct an extreme element μ of RP_1 such that the closed support of μ is all of T^2. We also construct an RP-measure which is not an extreme point, but which belongs to a proper weak* closed face of RP_1, is absolutely continuous with respect to Haar measure, and satisfies a certain necessary condition on extreme elements of RP_1.

The well known theorem of Herglotz asserts that, if u is a positive harmonic function on the open unit disk D, then there is a unique positive Borel measure μ on the unit circle T such that

$$u(z) = \int_T P_z(x)d\mu(x), \tag{1}$$

where $P_z(x)$ denotes the Poisson kernel $\operatorname{Re}(x + z)/(x - z)$. An equivalent way to state Herglotz's theorem is the following. Let f be holomorphic in D, have positive real part and satisfy $f(0) = 1$. Then there is a unique probability measure μ_f on T such that

$$\operatorname{Re} f(z) = \int_T P_z(x)d\mu_f(x). \tag{2}$$

It is clear from (2) that the correspondence $M_1(f) = \mu_f$ is a bijection between the convex sets $P(T)$ = Borel probability measures on T and

$$\mathscr{P}_1 = \{f \mid f \text{ holomorphic in } D, \operatorname{Re} f > 0, f(0) = 1\}.$$

Also, if $\mathscr{P}_1$ and $P(T)$ are equipped with the topology of uniform convergence on compacta and the weak* topology respectively, then M_1 is continuous. Moreover, M_1 is affine, i.e., it preserves convex combinations. It follows that

$$M_1(\text{ex } \mathscr{P}_1) = \text{ex } P(T),$$

Received by the editors on March 21, 1984 and in revised form on September 18, 1984.