ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 1, Winter 1986

LOCALLY INJECTIVE TORSION MODULES

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ABSTRACT. Let R be a commutative ring and $\mathscr{F}$ a Gabriel topology of R. We discuss the R's satisfying the condition that for all $\mathscr{F}$-torsion R-modules T, T is $\mathscr{F}$-injective if and only if T is locally $\mathscr{F}$-injective. With one interpretation of locally $\mathscr{F}$-injective, this characterizes the $\mathscr{F}$-local R's. With another interpretation of locally $\mathscr{F}$-injective, every $\mathscr{F}$-local R has this property, but not conversely.

All rings considered will be commutative rings and R will always denote a ring. Concerning torsion theories, we follow mainly the notation from the B. Stenström text [7]. Our point of view will be mostly in terms of Gabriel topologies. Use spec$R$ for the set of all prime ideals of R and mspec$R$ for the set of all maximal ideals of R. If I is an ideal of R, then define $\text{mspec}(I) = \{M \in \text{mspec}R: I \subset M\}$. If T is an R-module and $M \in \text{mspec}R$, then define $T(M) = \{x \in T: \text{mspec}\ (\text{Ann}_R(x)) \subset \{M\}\} = \{0\} \bigcup \{x \in T: \text{mspec}\ (Ann_R(x)) = \{M\}\}$. Clearly $T(M)$ is then an R-submodule of T. For $\mathscr{F}$ a Gabriel topology of R, then R is $\mathscr{F}$-local if (1.) $|\text{mspec}(I)| < \infty$ for all $I \in \mathscr{F}$, and (2.) $|\text{mspec}(P)| = 1$ for all $P \in \mathscr{F} \bigcap \text{spec}R$. Then for $\mathscr{F}$ a Gabriel topology of R, the following three conditions are equivalent: (1.) R is $\mathscr{F}$-local, (2.) $T = \bigoplus_{M \in \text{mspec}R} T(M)$ for all $\mathscr{F}$-torsion R-modules T, and (3.) $T \cong \bigoplus_{M \in \text{mspec}R} T_M$ for all $\mathscr{F}$-torsion R-modules T [**2**, Theorem 1.2]. See [**2**] for a general discussion and the history of the $\mathscr{F}$-local concept.

We introduce the local Gabriel topologies $\mathscr{F}\{M\}$ along with a few observations. If $\mathscr{F}$ is a Gabriel topology of R and $M \in \text{mspec}R$, then $\mathscr{F}\{M\} = \{I \in \mathscr{F}: \text{mspec}(I) \subset \{M\}\}$. For $P \in \text{spec}R$, let $\mathscr{F}(P) = \{I: I$ is an ideal of R and $I \not\subset P\}$. Then $\mathscr{F}(P)$ is a Gabriel topology of R. Since $\mathscr{F}\{M\} = \mathscr{F} \bigcap (\bigcap \{\mathscr{F}(P): P \in \text{mspec}R - \{M\}\})$, and the intersection of Gabriel topologies is a Gabriel topology, one infers that $\mathscr{F}\{M\}$ is a Gabriel topology of R. Note that if T is an R-module and $M \in \text{mspec}R$,

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AMS (MOS) Subject Classification (1980): Primary 13C11, 13B30, 13D30; Secondary 13C12.

Key words and phrases: Commutative ring, Gabriel topology, $\mathscr{F}$-injective R-module, $\mathscr{F}$-torsion R-module, $\mathscr{F}$-local commutative ring.

Received by the editors on September 14, 1983 and in revised form on April 6, 1984.