ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 3, Summer 1985

APPROXIMATION IN THE MEAN BY POLYNOMIALS

J. A. CIMA AND A. MATHESON*

ABSTRACT A new condition on crescent shaped regions Ω is given which insures that polynomials are dense in the Bergman spaces $A^p(\Omega)$. The relation of this condition to the classical results is described. Finally, a characterization of the dual of $A^p(\Omega)$ $(1 < p < \infty)$ is given for the region Ω bounded by two internally tangent circles.

1. Throughout this paper Ω will denote an open region of the complex plane $\mathbf{C}$ and σ will denote two-dimensional Lebesque measure. For a fixed Ω, $L^p(\Omega)$ denotes the usual space of Lebesgue-measurable, p-integrable functions on Ω, and $A^p(\Omega)$ will denote the space of functions which are both analytic on Ω and belong to $L^p(\Omega)$. Of course $A^p(\Omega)$ is a closed subspace of $L^p(\Omega)$. Finally, $H^p(\Omega)$ will denote the closure of the polynomials in $L^p(\Omega)$, so $H^p(\Omega)$ is a closed subspace of $A^p(\Omega)$.

2. The results of §3 and §4 come under the broad topic of approximation in the mean by polynomials on subdomains of the complex plane. The domains considered in this paper will be restricted to crescents lying between two internally tangent circles with one multiple boundary point. Much of the older work in this area appeared in the Soviet literature. The following theorem gives the flavor of the classical theory. The sufficiency was established by M. M. Dzrbasjan and the necessity by A. L. Saginjan.

THEOREM. *Let Ω be a crescent with multiple boundary point at the origin such that Ω is situated between the two circles $|z-1| = 1$ and $|z-(1/2)| = 1/2$. Denote by $\ell(r)$ the linear measure of $(|z| = r) \cap \Omega$ and assume that $r(\ell'(r)/\ell(r)) \uparrow +\infty$ as $r \downarrow 0$. Then in order for $H^p(\Omega) = A^p(\Omega)$ for any p it is necessary and sufficient that*

$$\int_0 \log \ell(r) dr = -\infty.$$

A survey paper by S. N. Mergelyan [7] describes the state of the art in

---

Received by the editors April 27, 1984.

* This work was carried out while the second author was visiting the University of North Carolina at Chapel Hill. He would like to thank that institution for the hospitality extended him during his stay.