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ON THE BETWEENNESS CONDITION OF ROLLE'S THEOREM

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1. Introduction. Rolle's theorem, in its simplest form, when applied to real polynomials $f(x)$, states that (1) between any two consecutive real zeros of $f(x)$ there is an odd number of zeros of the derivative, $Df(x)$; and consequently, (2) the polynomial $Df(x)$ has no more nonreal zeros than $f(x)$ has. Generalizations of this second property have been explored in [**1**, **2**]. Let $\Gamma = \{\gamma_k\}$ be a sequence of real numbers and for an arbitrary real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ define

$$\Gamma[f(x)] = \sum_{k=0}^{n} a_k \gamma_k x^k. \tag{1.1}$$

We recall that a sequence $\Gamma = \{\gamma_k\}$ of real numbers is called a multiplier sequence of the first kind if Γ takes every real polynomial $f(x)$ which has only real zeros into a polynomial $\Gamma[f(x)]$ (defined by (1.1)) of the same class. (For the various properties of multiplier sequences of the first kind we refer the reader to Pólya and Schur [**7**], Obreschkoff [**6**] and Craven and Csordas [**1**, **2**]). The relationship between these sequences and Rolle's theorem is suggested by the fact that for the multiplier sequence $\Gamma = \{0, 1, 2, \ldots\}$, we have $\Gamma[f(x)] = xf'(x)$.

The purpose of this paper is to generalize the betweenness condition of Rolle's theorem. We shall show (Corollary 2.4) that the class of linear transformations Γ, defined by (1.1), which satisfy the betweeness property (see Definition 2.1) is precisely the class of nonconstant arithmetic sequences, all of whose terms have the same sign. Our main theorem is a quantitative result on the location of the zero of $\Gamma[f(x)]$ between two consecutive real zeros a and b of f. This result gives the best possible bounds for the zero of $\Gamma[f(x)]$ depending only on Γ, a, b and the degree of f. This generalizes an old theorem of Laguerre for derivatives [**6**, p. 121] and its subsequent extension to a larger class of polynomials by Nagy [**5**]. We conclude the paper with some open problems.

2. The betweenness property. The precise formulation of the betweenness property is as follows.

DEFINITION 2.1. Let $f(x)$ be an arbitrary real polynomial. A real se-

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