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ON AN ERROR TERM OF LANDAU - II

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In memory of Ernst Straus and Robert A. Smith

ABSTRACT. Let $\phi(n)$ denote the Euler totient function. In 1900, E. Landau proved that $\sum_{n\leq x} 1/\phi(n) = A(\log x + B) + E_0(x)$ where $A > 0$ and B are constants and $E_0(x) = O(\log x/x)$. In an earlier paper, we sharpened this result and in the present paper prove, in particular, some average and Ω-type theorems for the error function $E_0(x)$.

1. Introduction. Let $\phi(n)$ denote the Euler totient function defined to be the number of positive integers $\leqq n$ and prime to n. We write

$$\sum_{n\leqq x} 1/\phi(n) = A(\log x + B) + E_0(x) \tag{1.1}$$

and

$$\sum_{n\leqq x} n/\phi(n) = Ax - \frac{1}{2}\log x + E_1(x) \tag{1.2}$$

where

$$A = \frac{315\zeta(3)}{2\pi^4} \text{ and } B = \gamma - \sum_p \frac{\log p}{p^2 - p + 1}. \tag{1.3}$$

Here ζ denotes the Riemann zeta function, γ denotes the Euler-Mascheroni constant and the sum on the extreme right of (1.3) extends over all primes p. In 1900, E. Landau (cf. [3, p. 184]) proved that, as $x \to \infty$,

$$E_0(x) = 0\ (\log x/x). \tag{1.4}$$

A systematic study of the error function $E_0(x)$ does not appear to have been made by later authors. In an earlier paper [5], using a theorem of A. Walfisz based on Weyl's inequality, we were able to improve upon (1.4) by proving that, as $x \to \infty$,

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